AIP Proceedings

Clocking convergence to Arnold tongues - The H-rank approach

Mantas Landauskas and Minvydas Ragulskis

Citation: AIP Conference Proceedings **1558**, 2457 (2013); doi: 10.1063/1.4826038
View online: http://dx.doi.org/10.1063/1.4826038
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1558?ver=pdfcov
Published by the AIP Publishing

This article is copyrighted as indicated in the abstract. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 78.57.136.187 On: Mon, 25 Nov 2013 08:04:01

Clocking Convergence to Arnold Tongues - The *H*-rank Approach

Mantas Landauskas and Minvydas Ragulskis

Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Kaunas University of Technology, Studentu 50-222, Kaunas, LT-51368, Lithuania

Abstract. Computational techniques based on ranks of Hankel matrices (H-ranks) is used to study the convergence to Arnold tongues in the circle map. It appears that the process of convergence to the phase-locked mode of the discrete stationary attractor is far from being trivial. Figures of pseudoranks of Hankel matrices constructed from transient solutions of the circle map carry important physical information about complex nonlinear processes and are also beautiful from the aesthetical point of view.

Keywords: Hankel matrix, Circle map
PACS: 89.70.Eg, 89.90.+n, 89.75.Kd

INTRODUCTION

Clocking convergence is an important tool for investigating various aspects of iterative nonlinear maps. The rate of convergence to the critical attractor when an ensemble of initial conditions is uniformly spread over the entire phase space may provide the insight into the fractal nature and the scale invariance of the dynamical attractor [1, 2]. Numerical convergence of the discrete logistic map gauged with a finite computational accuracy is investigated in [3] where forward iterations are used to identify self-similar patterns in the region before the onset to chaos. An alternative technique based on the concept of the H-rank is proposed in [4] for clocking the convergence of iterative chaotic maps.

The main objective of this paper is to show that the concept of the H-rank can be effectively used for the investigation of convergence properties of the circle map. We will use the discrete iterative circle map to illustrate the process of convergence to stationary states. The circle map is a paradigmatic model of a nonlinear iterative dynamical system used to study the dynamical behavior of a beating heart [5]. We will show that the study of the convergence rate to a periodic orbit of the circle map can produce beautiful and appealing patterns. Moreover, these graphical pictures contain important information on the stability of periodic orbits of the circle map.

THE ALGORITHM FOR THE COMPUTATION OF THE *H*-RANK

The concept of the H-rank of a sequence $(p_j; j = 0, 1, \ldots)$; $p_j \in \mathbb{R}$; has been introduced in [4]. The purpose of this section is to recall the concept of the H-rank of a solution of a discrete iterative map. Corresponding sequence of Hankel matrices reads:

$$H_n := \left(p_{i+j-2}\right)_{1\leq i,j\leq n} = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ \cdots & \cdots & \cdots & \cdots \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix}; n = 1, 2, \ldots.$$

The Hankel transform (the sequence of determinants of Hankel matrices) $(d_n; n = 0, 1, \ldots)$ reads:

$$d_n := det H_n; n = 1, 2, \ldots.$$

Definition. The sequence $(p_j; j = 0, 1, \ldots)$ has an H-rank $m \in \mathbb{Z}_0$; $m < +\infty$;

$$Hr(p_j; j = 0, 1, \ldots) = m;$$

11th International Conference of Numerical Analysis and Applied Mathematics 2013
AIP Conf. Proc. 1558, 2457-2460 (2013); doi: 10.1063/1.4826038
© 2013 AIP Publishing LLC 978-0-7354-1184-5/$30.00

This article is copyrighted as indicated in the abstract. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 78.57.136.187 On: Mon, 25 Nov 2013 08:04:01

if the sequence of determinants of Hankel matrices has the following structure:

$$(d_1, d_2, \ldots, d_m, 0, 0, \ldots)$$

where $d_m \neq 0$ and $d_{m+1} = d_{m+2} = \ldots = 0$.

It is admited that $Hr(0,0,0,\ldots) = 0$. Note that $Hr(p_0, \ldots, p_m, 0, 0, 0, \ldots) = m+1$, if only $p_m \neq 0$ for $m = 0, 1, 2, \ldots$.

VISUALIZATION OF THE PROCESS OF CONVERGENCE TO ARNOLD TONGUES

The circle map is represented by the one-dimensional iterative map:

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \cdot sin(2\pi\theta_n); \tag{1}$$

where θ_n value lies between 0 and 1 ($2\pi\theta_n$ is a polar angle); K is the coupling strength; Ω is the driving phase and $n = 0, 1, \ldots$. The circle map exhibits a phenomenon called phase locking if small to intermediate values of K ($0 < K < 1$) and certain values of Ω are considered. In a phase-locked region, the values θ_n advance as a rational multiple of n. The phase-locked regions in $\Omega - K$ parameter plane are called Arnold tongues [6].

We will use the H-rank as the computational tool for the reconstruction of Arnold tongues. We compute H-ranks in the region $0 \leq \Omega \leq 1$ and $0 \leq K \leq \pi$. For every pair of Ω and K we start the iterative process and construct the sequence $\{\theta_j\}$; $j = 0, 1, \ldots$; the initial condition θ_0 is set to 0.5; and calculate the H-rank of that sequence. As shown in [4], the H-rank of a chaotic sequence does not exist (the H-rank tends to infinity then). Therefore we set the upper limit for the H-rank $\overline{m} = 30$. If the sequence of determinants does not vanish until $m = 30$ we terminate the process assuming that $Hr\{p_j\} = \overline{m}$; $j = 0, 1, \ldots$. The results are shown in Figure 1(f). The more elements of the sequence ($\theta_j; j = 0, 1, \ldots$); are considered (leading to possible higher H-ranks of the sequence) the more resulting picture is alike to the well-known shape of Arnold tongues in the circle map [7].

THE COMPUTATION OF PSEUDORANKS

In order to determine the rank of the sequence one needs to find such matrix dimension ($\overline{m}+1$) that the determinant of the Hankel matrix is equal to zero. In practice it is sufficient to compute determinants up to a certain precision, like the machine epsilon. Calculating a determinant of a square real matrix requires a fair amount of computer resources if the dimension of a matrix is large. Moreover, the determinant, though being a conventional notion theoretically, rarely finds a useful role in numerical algorithms [8].

Plotting phase diagrams of H-ranks requires massive computations of determinants of Hankel matrices. Thus, instead of using a standard straightforward function *det* in MATLAB we use C++ and the LAPACK package to perform the computation of determinants of Hankel matrices. And though LAPACK can be considered as the state-of-the-art in linear algebra, it does not have a standard subroutine for the computation of the determinant. Instead, we perform the standard PLU decomposition of a matrix into the lower triangular matrix L (having ones on the main diagonal), the upper triangular matrix U (the absolute value of the determinant of the original matrix equals to the product of elements on the main diagonal of U) and the permutation matrix P. The number of permutations determines the sign of the determinant of the original matrix. But since we are interested in the absolute value of the determinant only, it suffices to compute the product of diagonal elements of U (an alternative approach could be counting the number of non-zero diagonal elements). We continue the computation of determinants as the product of diagonal elements of the matrix U until $|\det H_{\overline{m}+1}| < \varepsilon$. In this respect our computations reveal not the rank, but the pseudorank of a sequence.

The combination of the speed of C++ in performing loops (opposite to MATLAB) and the mathematical precision of LAPACK resulted in significantly faster formation of images of H-ranks in various phase planes. It can be noted that final visualization is performed using the functionality of MATLAB graphical functions.

The selection of a particular value of ε requires additional attention. As mentioned previously, the structure of Arnold tongues in the circle map is well-known. We perform the computation of pseudoranks for different initial conditions ($0 \leq \Omega \leq 1$, $0 \leq K \leq \pi$ and $\theta_0 = 0.5$) for different ε. Results are illustrated in Figure 1. The evolution of interesting patterns of pseudoranks can be observed as the value of ε is decreased (note that the maximum rank in colorbars is detected automatically and depends on ε).

This article is copyrighted as indicated in the abstract. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 78.57.136.187 On: Mon, 25 Nov 2013 08:04:01

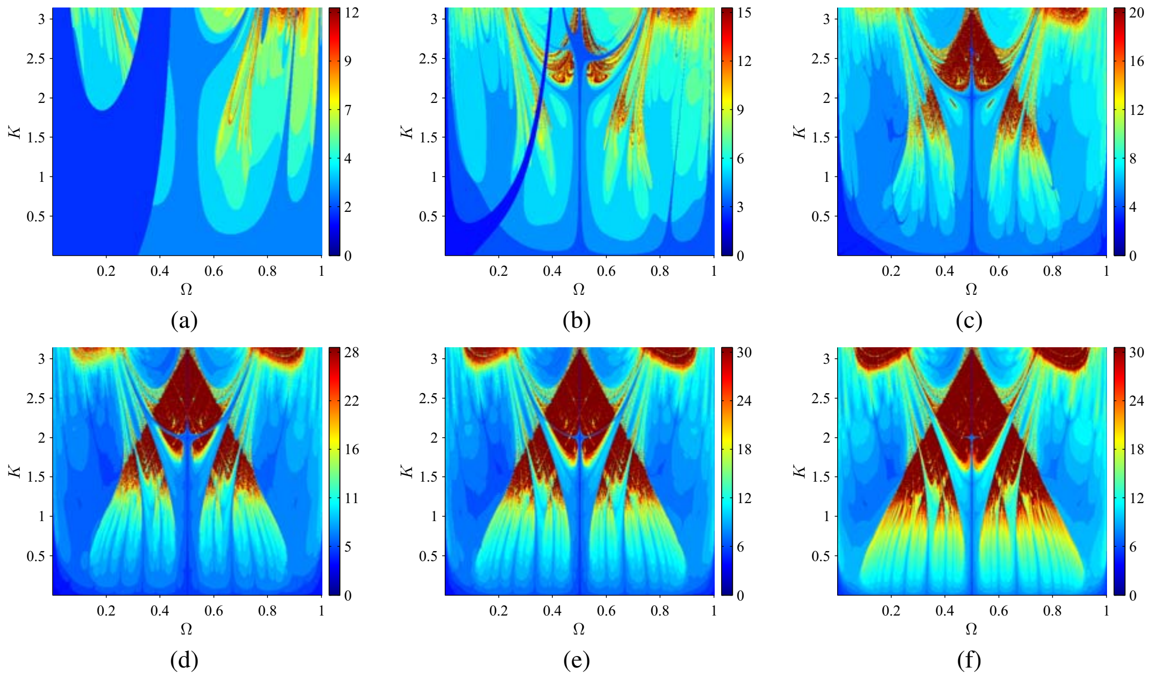

FIGURE 1. Maps of pseudoranks for different initial conditions ($0 \leq \Omega \leq 1$, $0 \leq K \leq \pi$ and $\theta_0 = 0.5$) at (a): $\varepsilon = 10^{-1}$; (b): $\varepsilon = 10^{-2}$; (c): $\varepsilon = 10^{-4}$; (d): $\varepsilon = 10^{-8}$; (e): $\varepsilon = 10^{-12}$; (f): $\varepsilon = 10^{-20}$.

A naked eye cannot see principal differences between Figure 1(e) and Figure 1(f). At this point we can fix the value of ε and use it for the construction of maps of pseudoranks. But the particular selection must be valid. In order to achieve this we construct the graph representing the absolute root mean square difference E between consecutive maps of pseudoranks in Figure 1. Let us denote $Hr_1(i,j)$ the value of the pseudorank at the i-th row and the j-th column of the map of pseudoranks computed at ε_1 (analogously $Hr_2(i,j)$ is the pseudorank at ε_2). Then, the difference E is defined as:

$$E(\varepsilon_2) = \sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(Hr_1(i,j) - Hr_2(i,j)\right)^2}$$

where m is the number of rows and n is the number of columns in maps of pseudoranks. The relationship $E(\varepsilon)$ is shown in Figure 2. It can be clearly seen that maps of pseudoranks do not change considerably beyond $\varepsilon = 10^{-25}$ and we reach the limit of accuracy of double floating point arithmetics.

CONCLUDING REMARKS

The existence of Arnold tongues in the circle map is known for already more than five decades ago [9]. There exist different computational techniques for the visualization of Arnold tongues. The universal algorithm for the identification of Arnold tongues is based on two simple steps. At first, the system must be iterated far away from initial conditions until all transient processed cease down. Secondly, one must identify the effect of the phase locking in the discrete stationary attractor. Different modes of the phase locking are then visualized by different colors.

The method discussed also consists of two steps and can be used for visualizing Arnold tongues themselves. Firstly the system is iterated for a predetermined number of steps from initial conditions. Then one do not need to search for the effect of the phase locking. Then simple computation of the H-rank of the stationary signal is performed and Arnold tongues occur in the phase plot of pseudoranks. Thus the quality of phase diagrams is crucial whenever the manipulation or control of quasiperiodic nonlinear systems would be considered.

This article is copyrighted as indicated in the abstract. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 78.57.136.187 On: Mon, 25 Nov 2013 08:04:01

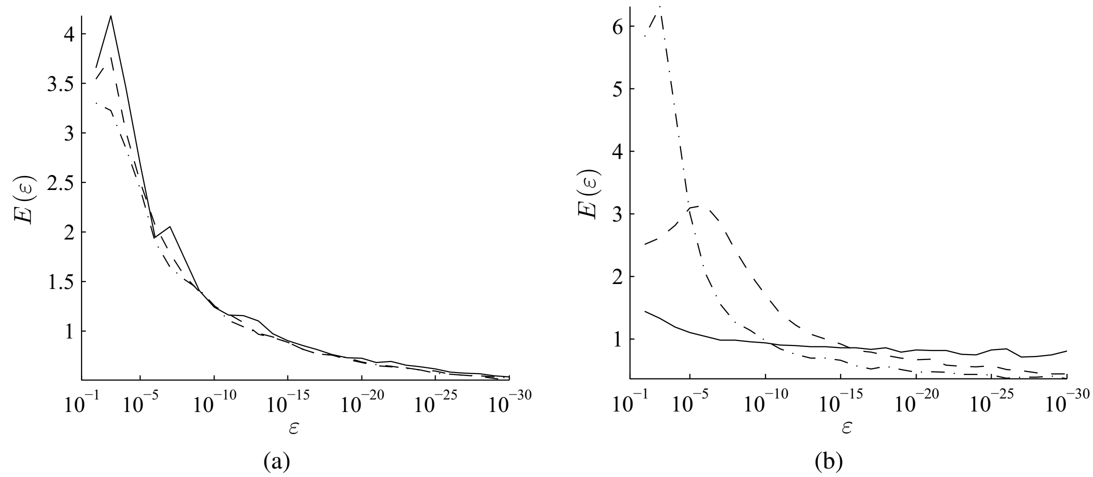

FIGURE 2. The relationship between the absolute root mean square difference E and the ε at $m = 30$. Calculations performed using parameter plane $\Omega - K$ (a) (the solid line represents the variation of E at $\theta_0 = 0.25$; the dashed line - at $\theta_0 = 0.5$; the dotted and dashed line - at $\theta_0 = 0.75$) and $\Omega - \theta_0$ (b) (the solid line represents the variation of E at $K = 0.25\pi$; the dashed line - at $K = 0.5\pi$; the dotted and dashed line - at $K = 0.75\pi$).

A much more interesting question is about the convergence properties of the circle map to Arnold tongues. It has been shown previously that pseudoranks of transient processes may reveal important physical information about the properties of a discrete system. For example, it has been shown in [6] that one can observe the stable, the unstable manifold and the manifold of nonasymptotic convergence in the plotted phase diagrams of the logistic map. In this paper the H-rank of the transient processes of the circle map was used for the visualization of the rate of convergence to the Arnold tongues. Optimal selection of the value for ε was considered.

ACKNOWLEDGEMENTS

Financial support from the Lithuanian Science Council under project No. MIP-100/2012 is acknowledged.

REFERENCES

1. F. A. B. F. de Moura, U. Tirnakli and M. L. Lyra, *Physical Review E* **62**, 6361–6365 2000.
2. R. Tonelli and M. Coraddu, *The European Physical Journal B* **50**, 355–359 (2006).
3. C. L. Bresten and Jae-Hun Jung, *ommun Nonlinear Sci Numer Simul* **14**, 3076–3088 (2009).
4. M. Ragulskis and Z. Navickas, *Commun Nonlinear Sci Numer Simul* **16**, 2894–2906 (2011).
5. L. Glass, M. R. Guevara, A. Shrier and R. Perez, *Physica 7D* **7**, 89–101 (1983).
6. P. L. Boyland, *Communications in Mathematical Physics* **106**, 353–381 (1986).
7. F. Schilder and B. B. Peckham, *Journal of Computational Physics* **220**, 932–951 (2007).
8. L. N. Trefethen, *Numerical linear algebra*, SIAM (1997).
9. V. I. Arnold, *Izv. Akad. Nauk SSSR Ser. Mat.* **25**, 21–86 (1961).

This article is copyrighted as indicated in the abstract. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 78.57.136.187 On: Mon, 25 Nov 2013 08:04:01