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Generalization of Exp-function and other standard function methods

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A B S T R A C T

Exp-function and other standard function methods for solving nonlinear differential equations are generalized in this paper. An analytical criterion determining if a solution can be expressed in a form comprising standard functions is derived. New computational algorithm for automatic identification of the structure of the solution is constructed. The algorithm provides information if the solution can be expressed as a sum of standard functions, a ratio of sums of standard functions, or even a more complex algebraic form involving standard functions. Several examples are used to illustrate the proposed concept.

1. Introduction

The Exp-function method was proposed in 2006 by He and Wu [1] to seek solitary solutions, periodic solutions and compacton-like solutions of nonlinear differential equations. It has been demonstrated that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving high-dimensional nonlinear evolutions in mathematical physics. The Exp-function method has been exploited for the determination of exact solutions of many nonlinear differential equations [2–5]; we give only a few of many references available.

Several alternative modifications of the Exp-function method have been developed. The tanh, extended tanh, improved tanh and generalized tanh function methods [6–9], sech and rational Exp-function method [10], the modified simplest equation method [11] and many similar standard function methods are also successfully used for the construction of solutions of nonlinear differential equations.

An analytical criterion determining if a solution of a differential equation can be expressed in an analytical form comprising exponential functions is developed in [12]. The employment of this criterion does not only give an answer to the above-stated question but gives the structure of the solution so that one does not have to guess what the form of the solution is. The load of symbolic calculations is brought before the structure of the solution is identified. This is in contrary to the Exp-function type methods where the structure of the solution is first guessed, and then symbolic calculations are exploited for the identification of parameters.

The object of this paper is to develop and generalize Exp-function type methods. We seek three objectives in this process:

(i) To investigate possibilities to express solutions of differential equations in forms comprising not only exponential, hyperbolic tangent functions, but other standard functions (for example radical functions). The objective is to form a class of standard functions which could be used to express solutions of a widest possible class of differential equations.

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(ii) To develop a convenient computational algorithm for the identification of the structure of the solution. These algorithms must eliminate the need for guessing how the structure of the solution looks like. In this sense our approach is an alternative (and much more general) method compared to Exp-function type methods where the structure of the solution is first guessed, and only then symbolic computations are used to calculate appropriate coefficients. Our algorithm identifies the structure of the solution automatically and provides information on how the solution is expressed in a form comprising a standard function (if it is a sum of standard functions, a ratio of sums of standard functions, or even a more complex form involving standard functions).

(iii) It is well known that eigenvalues and eigenvectors play a central role in analysis of linear differential equations. It would be of high interest to find analogies of eigenvalues and eigenvectors for nonlinear differential equations and dynamical systems. We propose an insight into nonlinear dynamical systems through the definition of the H-rank, the Hankel characteristic equation and its solutions.

2. Preliminary definitions

2.1. Functions and their extensions

Definition 1. A function $f(x;s_0,\dots,s_{n-1})$, where

$$f := f(x;s_0,\dots,s_{n-1}) = \sum_{j=0}^{+\infty} p_j(s_0,\dots,s_{n-1})\frac{(x-c)^j}{j!};\ |x-c| < R_f \tag{1}$$

is called an extended function if its domain is extended from its region of convergence $|x-c| < R_f$ into a wider region (the real axis or the complex plane) with the exception of a limited number of special singular points $x_1,\dots,x_m$ where $\lim_{x\to x_i}|f(x)| = +\infty;\ i=1,2,\dots,m$; where $x,c \in R; \{s_0,s_1,\dots,s_{n-1}\}$ is a finite set of parameters; n is the number of parameters; $s_0,s_1,\dots,s_{n-1} \in C$; complex functions $p_j(s_0,\dots,s_{n-1})$ are polynomials, ratios of polynomials or radicals of polynomials.

$F_{x;s_0,\dots,s_{n-1}}$ is denoted as a set of extended functions.

Definition 2. A function

$$y(x) = \sum_{j=0}^{+\infty} \frac{q_j}{j!}x^j, \tag{2}$$

where $q_j;\ q_0 = 1;\ q_j = \prod_{j=0}^{j-1}(a+bk);\ q_j \neq 0;\ j=1,2,\dots$ are coefficients of the expansion, is called a standard function.

We will use several standard functions for the construction of generalized solutions of differential equations (we analyze these solutions only on the real x-axis):

$$\begin{aligned} y_1(x) &= \sum_{j=0}^{+\infty}\frac{x^j}{j!};\ y_1(x) = \exp(x);\ x\in R;\ q_j = 1,\\ y_2(x) &= \sum_{j=0}^{+\infty} x^j;\ |x|<1;\ y_2(x) = \frac{1}{1-x};\ x\neq 1;\ q_j = j!,\\ y_3(x) &= \sum_{j=0}^{+\infty}\frac{(2j-1)!!}{j!}x^j;\ (-1)!!=1;\ 1!!=1;\ 3!!=1\cdots 3;\dots;|x|<\frac{1}{2};\ y_3(x) = \frac{1}{\sqrt{1-2x}};\ x<\frac{1}{2};\ q_j = (2j-1)!!. \end{aligned} \tag{3}$$

Necessary order derivatives of the described functions do exist and this fact will be not stressed further on.

2.2. Differential and multiplicative operators

Definition 3. The generalized differential operator is defined as [13]:

$$D := Q_0 D_{s_0} + \cdots + Q_{n-1}D_{s_{n-1}}, \tag{4}$$

where $Q_0,\dots,Q_{n-1} \in F_{s_0,\dots,s_{n-1}};\ F_{s_0,\dots,s_{n-1}}$ is a set of extended functions which do not depend on $x; F_{s_0,\dots,s_{n-1}} \subset F_{x;s_0,\dots,s_{n-1}}; D_{s_j}$ are conventional linear differential operators (partial differentiation is performed in respect of a parameter s_j); $D^0_{s_j} := \mathbf{1};\ j=0,1,\dots,n-1;\ \mathbf{1}$ is an identical linear operator.

Usual properties of differentiation hold for the generalized differential operator [13]:

$$\begin{aligned} &D\left(\sum_{j=1}^{m} c_j f_j\right) = \sum_{j=1}^{m} c_j \cdot Df_j; \\ &D^n(f_k \cdot f_l) = \sum_{j=0}^{n} \binom{n}{j} \left(D^j f_k\right)\left(D^{n-j} f_l\right); \\ &Df^m = mf^{m-1}Df; \\ &D\frac{f_k}{f_l} = \frac{(Df_k)f_l - f_k(Df_l)}{f_l^2}; \\ &\ldots \\ &m,n = 0,1,\ldots;\ c_j \in \mathbf{C};\ j = 1,2,\ldots,m;\ k,l = 1,2,\ldots,n;\ f, f_k, f_l \in F_{x;s_0,\ldots,s_{n-1}}. \end{aligned} \tag{5}$$

It can be noted that the generalized differential operator sometimes is referred to as a vector field [15].

Definition 4. The multiplicative operator [13] is a linear operator:

$$G := G_D := \sum_{j=0}^{+\infty} \frac{x^j}{j!} D^j. \tag{6}$$

It can be noted that the multiplicative operator is denoted as the exponential operator $G_D := e^{xD}$ in [15,16].

We will enumerate several properties of the multiplicative operator:

$$\begin{aligned} &G\sum_{j=1}^{m} c_j f_j = \sum_{j=1}^{m} c_j G f_j; \\ &Gf(c; s_0,\ldots,s_{n-1}) = f(Gc; Gs_0,\ldots,Gs_{n-1}); \\ &G\frac{f_k(c; s_0,\ldots,s_{n-1})}{f_l(c; s_0,\ldots,s_{n-1})} = \frac{f_k(Gc; Gs_0,\ldots,Gs_{n-1})}{f_l(Gc; Gs_0,\ldots,Gs_{n-1})}; \end{aligned} \tag{7}$$

where $c_j \in \mathbf{C};\ j = 1,2,\ldots,m;\ f(x; s_0,\ldots,s_{n-1}),\ f_k(x; s_0,\ldots,s_{n-1}),\ f_l(x; s_0,\ldots,s_{n-1}) \in F_{x;s_0,\ldots,s_{n-1}}$.

For example,

$$e^{xD_c}c^n = (x+c)^n;\ e^{xD_c}f(c; s_0,\ldots,s_{n-1}) = f(c+x; s_0,\ldots,s_{n-1}). \tag{8}$$

2.3. Operator method for solving ordinary differential equations

2.3.1. First order ordinary differential equations, the initial value problem

The exact solution of a differential equation

$$y'_x = P_1(x,y);\ y(x_0; s) = s;\ x_0 \in R;\ y \in F_{x;s_0} \tag{9}$$

reads [13]:

$$y = \sum_{j=0}^{+\infty} \frac{(x-x_0)^j}{j!} \left((D_c + P_1(c,s)D_s)^j s\right)\Big|_{c=x_0}, \tag{10}$$

where the superscript denotes a full derivative and the subscript denotes the variable of differentiation. It can be noted that the solution is expressed in a form of an infinite order polynomial.

2.3.2. nth order ordinary differential equations, the initial value problem

The exact solution of an initial value problem

$$\begin{aligned} &y_x^{(n)} = P_n\left(x, y, y'_x, \ldots, y_x^{(n-1)}\right); \\ &y_x^{(j)}(x; s_0, s_1, \ldots, s_{n-1})\big|_{x=x_0} = s_j;\ x_0 \in R; P_n(x, s_0, s_1, \ldots, s_{n-1}) \in F_{x;s_0,\ldots,s_{n-1}} \end{aligned} \tag{11}$$

reads:

$$y = \sum_{j=0}^{+\infty} \frac{(x-x_0)^j}{j!} \left(D^j s_0\right)\Big|_{c=x_0}; \tag{12}$$

where

$$D := D_c + s_1 D_{s_0} + s_2 D_{s_1} + \cdots + s_{n-1} D_{s_{n-2}} + P_n(c, s_0, s_1, \ldots, s_{n-1}) D_{s_{n-1}}. \tag{13}$$

Example 1. Let $D^{n+1}s_0 = 0$, but $D^n s_0 \neq 0$. Then, $D^{n+k}s_0 = 0$ for $k = 1, 2, \ldots$, and $y(x; s_0, s_1, \ldots, s_{n-1}) = \sum_{j=0}^{n} \frac{(x-x_0)^j}{j!} \left(D^j s_0\right)|_{c=x_0}$; meaning that the function $y(x; s_0, s_1, \ldots, s_{n-1})$ is an nth order polynomial of x. It can be noted that $\{cD^n s | c \in C\} \subseteq KerD$.

2.3.3. Systems of first order ordinary differential equations, the initial value problem

The exact solution of a system of differential equations

$$\begin{cases} z'_x = R(x, z, u, v); z(x_0; s_0, s_1, s_2) = s_0; \\ u'_x = P(x, z, u, v); u(x_0; s_0, s_1, s_2) = s_1; \\ v'_x = Q(x, z, u, v); v(x_0; s_0, s_1, s_2) = s_2; \end{cases} \tag{14}$$

reads:

$$\begin{aligned} z &= \sum_{j=0}^{+\infty} \frac{(x-x_0)^j}{j!} \left(D^j s_0\right)\Big|_{c=x_0}; \\ u &= \sum_{j=0}^{+\infty} \frac{(x-x_0)^j}{j!} \left(D^j s_1\right)\Big|_{c=x_0}; \\ v &= \sum_{j=0}^{+\infty} \frac{(x-x_0)^j}{j!} \left(D^j s_2\right)\Big|_{c=x_0}; \end{aligned} \tag{15}$$

where $D = D_c + R(c, s_0, s_1, s_2)D_{s_0} + P(c, s_0, s_1, s_2)D_{s_1} + Q(c, s_0, s_1, s_2)D_{s_2}$.

Example 2. The generalized differential operator for a system

$$\begin{cases} z'_x = 1; z(x_0; c, s) = c; \\ u'_x = \frac{u}{z}; u(x_0; c, s) = s; \end{cases} \tag{16}$$

reads: $D = D_c + \frac{s}{c} D_s$. Thus, $D^0 c = c$; $Dc = 1$; $D^0 s = s$; $Ds = \frac{s}{c}$; but $D^2 c = D^2 s = 0$. Thus, the expression on the solution (z, u) reads:

$$\begin{aligned} z &= c + (x - x_0); \\ u &= s + \frac{s}{c}(x - x_0). \end{aligned} \tag{17}$$

It can be noted that the system (16) is equivalent to the differential equation $xy' = y$ which general solution is

$$y = ax; \; \left(a = \frac{s}{c} \in \mathbf{C}\right). \tag{18}$$

2.4. The H-rank of a sequence

A Hankel matrix [14] for the sequence $(p_j;\ j = 0, 1, 2, \ldots)$ where $p_j \in F_{x;s_0,\ldots,s_{n-1}}$ or $p_j \in \mathbf{C}$ is defined as:

$$H_n := \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ & & \cdots & \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix}, \quad n = 1, 2, \ldots. \tag{19}$$

Definition 5. A sequence $(p_j;\ j \in Z_0)$ has an H-rank $Hr(p_j;\ j \in Z_0) = m;\ m \in Z_0, m < +\infty$ if the sequence of determinants of Hankel matrixes has the following form: $(d_1; d_2; \ldots; d_m; 0; 0; \ldots)$; where $d_m \neq 0; d_{m+1} = d_{m+2} = \cdots = 0$.

Definition 6. The length of a sequence $(p_j;\ j \in Z_0)$ is $L(p_j;\ j \in Z_0) = m;\ m \in Z_0$, if the sequence has the form $(p_0; p_1; \ldots; p_m; 0; 0; \ldots)$; where $p_m \neq 0; p_{m+1} = p_{m+2} = \cdots = 0$.

Corollary 1. *If $L(p_j; j \in Z_0) = m;\ m < +\infty$, then $Hr(p_j; j \in Z_0) = m$.*

Definition 7. The characteristic Hankel equation for a sequence $(p_j;\ j \in Z_0)$ which H-rank is equal to m is defined as:

$$\det \begin{bmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ & & \cdots & \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{bmatrix} = 0. \tag{20}$$

Expansion of the determinant in Eq. (20) yields an mth order algebraic equation for determination of roots of the characteristic equation:

$$\rho^m + A_{m-1}\rho^{m-1} + \cdots + A_1\rho + A_0 = 0. \tag{21}$$

Corollary 2. *The following statement holds true* [14]:*Let* $Hr(p_j;\ j \in Z_0) = m$ *and the multiplicity of roots* $\rho_1, \rho_2, \ldots, \rho_l$ *of the characteristic equation* (Eq. (21)) *is accordingly* $m_1, m_2, \ldots, m_l$; $\sum_{r=1}^{l} m_r = m$. *Then,*

$$p_j = \sum_{r=1}^{l}\sum_{k=0}^{m_r-1} \mu_{rk}\binom{j}{k}\rho_r^{j-k}, \tag{22}$$

where $\mu_{rk}, \rho_r \in C$.

The opposite statement also holds true. If Eq. (22) holds, then

$$Hr(p_j; j \in Z_0) = m_1 + m_2 + \cdots + m_l. \tag{23}$$

Definition 8. A sequence is entitled as an algebraic sequence if elements of that sequence $(p_j;\ j \in Z_0)$ are defined by Eq. (22). We assume that $\mu_{rk}\binom{j}{k}\rho_r^{j-k} = 0$ if $\binom{j}{k} = 0$ what is true when $0 \leqslant j < k$, where $\binom{j}{k} = \frac{j!}{k!(j-k)!}$.

Corollary 3. *In case when all roots of the characteristic equation are different*, Eq. (22) *obtains a more simple form:*

$$p_j = \sum_{r=1}^{m} \mu_r \rho_r^j. \tag{24}$$

It can be noted that coefficients μ_{rk} (or just μ_r) can be found solving the linear algebraic system of equations ($\rho_1, \rho_2, \ldots, \rho_l$ are already determined):

$$\sum_{r=1}^{l}\sum_{k=0}^{m_r-1} \binom{j}{k}\rho_r^{j-k}\mu_{rk} = p_j; \quad j = 0, 1, \ldots, m-1. \tag{25}$$

This system of equations always has the only solution [14].

It can be also noted that one can use Eq. (22) or Eq. (24) to calculate all elements of the series $(p_j;\ j \in Z_0)$ starting from p_{2m} if $Hr(p_j;\ j \in Z_0) = m$ and the first $2m$ elements of that series are known.

Example 3. Lets consider a sequence $(p_0; p_1; p_2; 0; 0; \ldots)$ where $p_2 \neq 0$ and $p_j = 0;\ j = 3, 4, \ldots$. Then, $\det H_3 = -p_2^3 \neq 0$; but $\det H_j = 0$ for $j = 4, 5, \ldots$. Thus, $Hr(p_0; p_1; p_2; 0; 0; \ldots) = 3$. It is clear that

$$\begin{vmatrix} p_0 & p_1 & p_2 & 0 \\ p_1 & p_2 & 0 & 0 \\ p_2 & 0 & 0 & 0 \\ 1 & \rho & \rho^2 & \rho^3 \end{vmatrix} = 0, \tag{26}$$

what yields $\rho^3 p_3 = 0;\ \rho_1 = \rho_2 = \rho_3 = 0$. Therefore, finally:

$$p_j = p_0 0^j + p_1\binom{j}{1}0^{j-1} + p_2\binom{j}{2}0^{j-2}; \quad j = 0, 1, 2, \ldots, \text{ where } 0^0 := 1;\ 0^1 = 0^2 = \cdots = 0. \tag{27}$$

2.5. Changing the independent variable of a differential equation

The independent variable of a differential equation can be changed in order to produce a more convenient form of the exact solution of a differential equation.

Let an invertible function is given by following equalities:

$$x := \varphi(z); \quad z = \psi(x). \tag{28}$$

Then following expressions hold:

$$\begin{aligned} z'_x &= \psi'_x = \frac{1}{\varphi'_z}; \\ z''_{xx} &= -\frac{1}{(\varphi'_z)^2}\varphi''_{zz} \cdot z'_x = -\frac{\varphi''_{zz}}{(\varphi'_z)^3}; \\ &\cdots. \end{aligned} \tag{29}$$

Also, for every function

$$y = y(x) = y(\varphi(z)) := \omega(z) = \omega(\psi(x)) = \omega \tag{30}$$

following equalities hold:

$$\begin{aligned} y'_x &= \omega'_z\big|_{z=\psi(x)} \cdot \psi'_x = \frac{1}{\varphi'_z} \cdot \omega'_z\big|_{z=\psi(x)}; \\ y''_{xx} &= \omega''_{zz}\big|_{z=\psi(x)} \cdot \left(\psi'_x\right)^2 + \omega'_z\big|_{z=\psi(x)} \cdot \psi''_{xx} = \frac{1}{\left(\varphi'_z\right)^2} \cdot \omega''_{zz}\big|_{z=\psi(x)} - \frac{\varphi''_{zz}}{\left(\varphi'_z\right)^3} \cdot \omega'_z\big|_{z=\psi(x)}; \\ &\cdots. \end{aligned} \tag{31}$$

Thus,

$$\begin{aligned} y'_x\big|_{x=\varphi(z)} &= \frac{1}{\varphi'_z} \cdot \omega'_z; \\ y''_{xx}\big|_{x=\varphi(z)} &= \frac{1}{\left(\varphi'_z\right)^2} \cdot \omega''_{zz} - \frac{\varphi''_{zz}}{\left(\varphi'_z\right)^3} \cdot \omega'_z; \\ &\cdots. \end{aligned} \tag{32}$$

Merging Eqs. (29)–(32) into one equality and using symbol τ to identify the change of variable in the differential equation yields:

$$\tau\left(x; y; y'_x; y''_{xx}; \ldots\right) := \left(x|_{x=\varphi(z)}; y|_{x=\varphi(z)}; y'_x|_{x=\varphi(z)}; y''_{xx}|_{x=\varphi(z)}; \cdots\right); \tag{33}$$

or

$$\tau\left(x; y; y'_x; y''_{xx}; \ldots\right) := \left(\varphi(z); \omega; \frac{1}{\varphi'_z} \cdot \omega'_z; \frac{1}{\left(\varphi'_z\right)^2} \cdot \omega''_{zz} - \frac{\varphi''_{zz}}{\left(\varphi'_z\right)^3} \cdot \omega'_z; \cdots\right). \tag{34}$$

For example, the image of a second order differential equation (Eq. (11) at $n = 2$) takes the following form after the variable change $x = \varphi(z)$:

$$\frac{1}{\left(\varphi'_z\right)^2} \omega''_{zz} - \frac{\varphi''_{zz}}{\left(\varphi'_z\right)^3} \omega'_z = P_2\left(\varphi(z); \omega, \frac{1}{\varphi'_z} \omega'_z\right). \tag{35}$$

Example 4. Several typical variable changes are listed below:

(i) Exponential variable change $\exp(\alpha x) = z$:

$$\tau\left(x; y; y'_x; y''_{xx}; \ldots\right) := \left(\frac{1}{\alpha} \ln z; \omega; \alpha z \omega'_z; \alpha^2\left(z^2 \omega''_{zz} + z\omega'_z\right); \ldots\right). \tag{36}$$

(ii) Logarithmic variable change $\alpha \ln x = z$:

$$\tau\left(x; y; y'_x; y''_{xx}; \ldots\right) := \left(\exp\left(\frac{z}{\alpha}\right); \omega; \exp\left(-\frac{z}{\alpha}\right)\omega'_z; \exp\left(-\frac{2z}{\alpha}\right)\left(\omega''_{zz} - \omega'_z\right); \ldots\right). \tag{37}$$

(iii) Symmetric variable change $\frac{\alpha}{x} = z$:

$$\tau\left(x; y; y'_x; y''_{xx}; \ldots\right) := \left(\frac{\alpha}{z}; \omega; -\frac{z^2}{\alpha} \omega'_z; \frac{2z^3}{\alpha^2} \omega'_z + \frac{z^4}{\alpha^2} \omega''_{zz}; \ldots\right). \tag{38}$$

where α is a parameter of the variable change; $\alpha \in R; \alpha \neq 0$.

3. Main theorems

We will prove one of the fundamental properties of the generalized differential operator. This property can be used to express a solution of a differential equation in a form comprising standard functions.

3.1. Theorem 1

The formulation of the Theorem.

(a) A generalized differential operator $D := Q_1 D_{s_1} + \cdots + Q_n D_{s_n};\ Q_1, \ldots, Q_n \in F_{s_1,\ldots,s_n}$ is used to calculate functions $p_j := D^j s;\ p_j \in F_{s_1,\ldots,s_n};\ j = 0, 1, \ldots$.
(b) A rule $q_0 := 1;\ q_{j+1} = \prod_{k=0}^{j}(a + bk);\ j = 0, 1, 2, \ldots$ is used to construct a sequence $(q_j;\ j \in Z_0);\ a, b \in \mathbf{C}$ are such constants that $q_j \neq 0$.

(A) Then, three following statements can have a sense (but can be individually true or not true):
(i) $p_j = q_j \sum_{r=1}^{m} \mu_r \lambda_r^j$, where $\lambda_k, \mu_k \in F_{s_1,\ldots,s_n};\ \lambda_k \neq \lambda_l$ and $\mu_k \neq 0$ when $k \neq l$ and $k, l = 1, 2, \ldots, m;\ m \in \mathbf{N}$ and is a fixed constant; $j = 0, 1, 2, \ldots$.
(ii) $Hr\left(\frac{1}{q_j} p_j;\ j \in Z_0\right) = n;\ n \in \mathbf{N}$ and is a fixed constant. Also, roots of Hankel characteristic equation $\rho_1, \rho_2, \ldots, \rho_n \in F_{s_1,\ldots,s_n}$ are all different.
(iii) There exists a set of functions $\gamma_1, \gamma_2, \ldots, \gamma_{\bar{n}}, \sigma_1, \sigma_2, \ldots, \sigma_{\bar{n}} \in F_{s_0,\ldots,s_{n-1}}$ ($\bar{n} \in \mathbf{N}$ and is a fixed constant) which satisfy following conditions:

$$\begin{aligned} &(a)\gamma_k \neq \gamma_l; \quad \sigma_k \neq 0;\\ &(b)\sigma_1 + \sigma_2 + \cdots + \sigma_{\bar{n}} = s;\\ &(c)D\sigma_k = \alpha\sigma_k\gamma_k;\\ &(d)D\gamma_k = \beta\gamma_k^2; \end{aligned} \tag{39}$$

where $k \neq l; k = 1, 2, \ldots, \bar{n}; \alpha, \beta \in \mathbf{C}$.
(B) Statements (i), (ii) and (iii) are equivalent.

Proof

1. The equivalency of statements (i) and (ii) follows from results proven in [14]. Also, $m = n$ and $\lambda_k = \rho_k$ when $k = 1, 2, \ldots, m$.
2. Lets assume that the statement (iii) holds true. Then, $p_0 = D^0 s = s = q_0 s$;

$$p_1 = D\sum_{r=1}^{\bar{n}} \sigma_r = \sum_{r=1}^{\bar{n}} D\sigma_r = \alpha \sum_{r=1}^{\bar{n}} \sigma_r\gamma_r = q_1 \sum_{r=1}^{\bar{n}} \sigma_r\gamma_r.$$

We make a proposition that $D^j s = p_j = \prod_{l=0}^{j-1}(\alpha + \beta l)\sum_{r=1}^{\bar{n}} \sigma_r \gamma_r^j;\ j \in \mathbf{N}$ and is a fixed constant. Then,

$$D^{j+1}s = Dp_j = \prod_{l=0}^{j-1}(\alpha + \beta l) \sum_{r=1}^{\bar{n}} \left((D\sigma_r)\gamma_r^j + \sigma_i\left(D\gamma_r^j\right)\right) = \prod_{l=0}^{j-1}(\alpha + \beta l) \sum_{r=1}^{\bar{n}} \left(\alpha\sigma_r\gamma_r^{j+1} + j\sigma_r\beta\gamma_r^{j+1}\right) = \prod_{l=0}^{j}(\alpha + \beta l) \sum_{r=1}^{\bar{n}} \sigma_r\gamma_r^{j+1} = p_{j+1}. \tag{40}$$

Thus, the statement (ii) (and the statement (i)) holds true.
3. Lets assume that statements (i) and (ii) hold true. Then,

$$D^{j+1}s = q_j D \sum_{r=1}^{m} \mu_r \lambda_r^j = q_j \sum_{r=1}^{m} \left(\lambda_r^j (D\mu_r) + j\mu_r\lambda_r^{j-1}(D\lambda_r)\right). \tag{41}$$

Thus, the following system of equations is produced:

$$\sum_{r=1}^{m} \left(\lambda_r^j (D\mu_r) + j\mu_r\lambda_r^{j-1}(D\lambda_r)\right) = \frac{1}{q_j} q_{j+1} \sum_{r=1}^{m} \mu_r \lambda_r^{j+1}; \tag{42}$$

which is satisfied if following identities hold: $D\mu_r = a\mu_r\lambda_r$ and $D\lambda_r = b\mu_r\lambda_r^2$. Thus, the system of equations has at least one solution. On the other hand, the extended Van-Der-Mond determinant of coefficients at unknowns $D\mu_r$ and $D\lambda_r$ (when j sweeps over any of natural numbers $0 \leqslant j_1 < j_2 < \cdots < j_{2m}$)

$$\begin{aligned} \Delta &= \begin{vmatrix} \lambda_1^{j_1} & \lambda_2^{j_1} & \cdots & \lambda_m^{j_1} & j_1\mu_1\lambda_1^{j_1-1} & j_1\mu_2\lambda_2^{j_1-1} & \cdots & j_1\mu_m\lambda_m^{j_1-1} \\ \lambda_1^{j_2} & \lambda_2^{j_2} & \cdots & \lambda_m^{j_2} & j_2\mu_1\lambda_1^{j_2-1} & j_2\mu_2\lambda_2^{j_2-1} & \cdots & j_2\mu_m\lambda_m^{j_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{j_{2m}} & \lambda_2^{j_{2m}} & \cdots & \lambda_m^{j_{2m}} & j_{2m}\mu_1\lambda_1^{j_{2m}-1} & j_{2m}\mu_2\lambda_2^{j_{2m}-1} & \cdots & j_{2m}\mu_m\lambda_m^{j_{2m}-1} \end{vmatrix} \\ &= \mu_1\mu_2 \cdot \ldots \cdot \mu_m \begin{vmatrix} \lambda_1^{j_1} & \lambda_2^{j_1} & \cdots & \lambda_m^{j_1} & j_1\lambda_1^{j_1-1} & j_1\lambda_2^{j_1-1} & \cdots & j_1\lambda_m^{j_1-1} \\ \lambda_1^{j_2} & \lambda_2^{j_2} & \cdots & \lambda_m^{j_2} & j_2\lambda_1^{j_2-1} & j_2\lambda_2^{j_2-1} & \cdots & j_2\lambda_m^{j_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{j_{2m}} & \lambda_2^{j_{2m}} & \cdots & \lambda_m^{j_{2m}} & j_{2m}\lambda_1^{j_{2m}-1} & j_{2m}\lambda_2^{j_{2m}-1} & \cdots & j_{2m}\lambda_m^{j_{2m}-1} \end{vmatrix} \end{aligned} \tag{43}$$

cannot be equivocally equal to zero. Thus, the solution of the system of equations is the only one. Finally, the validity of the statement (iii) follows from the validity of statements (i) and (ii) at $m=\bar{n};\ \lambda_k=\gamma_k;\ \mu_k=\sigma_k;\ k=1,2,\ldots,\ \bar{n};\ a=\alpha$ and $b=\beta$. □

It can be noted that Theorem 1 holds only when all roots of the Hankel characteristic equation are different.

Theorem 1 forms the theoretical background for the construction of general solutions of differential equations. Let a sequence $(D^js;\ j\in Z_0)$ satisfy equivalent statements (i), (ii) and (iii) of the Theorem 1. Then, a general solution $y=y(x;s_0,s_1;\ldots,s_{n-1})$ of a differential equation can be expressed in the following form:

$$y(x;s_0,s_1,\ldots,s_{n-1})=\sum_{j=0}^{+\infty}(q_j\sum_{r=1}^{m}\mu_r\lambda_r^j)\frac{(x-x_0)^j}{j!}=\sum_{r=1}^{m}\mu_r\sum_{j=0}^{+\infty}\frac{q_j}{j!}(\lambda_r(x-x_0))^j=\sum_{r=1}^{m}\mu_r f(\lambda_r(x-x_0)); \tag{44}$$

where $f(x)$ is a standard function defined in Definition 2.

3.2. Theorem 2

Roots of the Hankel characteristic equation coincide with roots of the standard characteristic equation $\lambda^m-a_{m-1}\lambda^{m-1}-\cdots-a_1\lambda-a_0=0$ for linear differential equations with constant coefficients.

Proof. The initial value problem reads:

$$\begin{aligned}&y_x^{(m)}=a_{m-1}y_x^{(m-1)}+\cdots+a_1y_x'+a_0y;y_x^{(r)}(x;s_0,\ldots,s_{m-1})|_{x=x_0}=s_r;\\&r=0,1,\ldots,m-1;\ a_0,a_1,\ldots,a_{m-1}\in R.\end{aligned} \tag{45}$$

We select $q_j=1;\ j=0,1,2,\ldots$.

Then, the generalized differential operator takes the following form (Eq. (13)):

$$D=s_1D_{s_0}+s_2D_{s_1}+\cdots+s_{m-1}D_{s_{m-2}}+(a_{m-1}s_{m-1}+\cdots+a_1s_1+a_0s_0)D_{s_{m-1}}. \tag{46}$$

It can be noted that $-a_0s_0-a_1Ds_0-\cdots-a_{m-1}D^{m-1}s_0+D^ms_0=0$ because $-a_0s_0-a_1s_1-\cdots-a_{m-1}s_{m-1}+a_{m-1}s_{m-1}+\cdots+a_1s_1+a_0s_0=0$. Thus, $-a_0D^js_0-a_1D^{j+1}s_0-\cdots-a_{m-1}D^{j+m-1}s_0+D^{j+m}s_0=0;\ j=0,1,2,\ldots$. The last equalities yield:

$$d_{m+1}=\det\begin{bmatrix}D^0s_0 & Ds_0 & \cdots & D^ms_0\\ Ds_0 & D^2s_0 & \cdots & D^{m+1}s_0\\ \cdots & \cdots & \cdots & \cdots\\ D^ms_0 & D^{m+1}s_0 & \cdots & D^{2m}s_0\end{bmatrix}=0. \tag{47}$$

But

$$d_m=\det\begin{bmatrix}D^0s_0 & \cdots & D^{m-1}s_0\\ \cdots & \cdots & \cdots\\ D^{m-1}s_0 & \cdots & D^{2m-2}s_0\end{bmatrix}\neq 0. \tag{48}$$

Thus, the sequence of determinants is $(d_1;d_2;\ldots;d_m;0;0;\ldots)$. Therefore, $Hr(D^js;\ j=0,1,2,\ldots)=m$. Moreover, $b=0$ because $q_j=1$. Next, one has to find roots of the Hankel characteristic equation $\rho_1,\rho_2,\ldots,\rho_m$.

If all roots are different, then $D^js_0=\sum_{r=1}^m\mu_r(s_0,s_1,\ldots,s_{m-1})\rho_r^j$. Thus, $D\mu_r=\mu_r\rho_r;D\rho_r=0;\ r=1,2,\ldots,m-1$. Finally,

$$y=\sum_{r=1}^{m}\mu_r(s_0,s_1,\ldots,s_{m-1})\exp(\rho_r(x-x_0)).\qquad\square \tag{49}$$

It can be noted that in general case not only roots but also their multiplicities of the Hankel characteristic equation coincide with roots of the standard characteristic equation if only the equation is a linear differential equation with constant coefficients.

3.3. The case when some roots of the Hankel characteristic equation are multiple

It is clear that Theorem 1 does not hold when some roots of the Hankel characteristic equation are multiple. In that case the structure of the solution becomes more complex and cannot be expressed a form comprising only standard functions (the independent variable also figures in the expression then). We will give a short discussion and an example though this is an object of the future research.

Let $D^js=q_j\sum_{r=1}^{l}\sum_{k=0}^{m_r-1}\mu_{rk}\binom{j}{k}\rho_r^{j-k}$. In other words, multiplicities of roots of the Hankel characteristic equation $\rho_1;\rho_2;\ldots;\rho_l$ are $m_1;m_2;\ldots;m_l$ accordingly. Then, Eq. (12) yields:

$$\sum_{j=0}^{+\infty} q_j \binom{j}{k} \rho_r^{j-k} \frac{(x-x_0)^j}{j!}\bigg|_{c=x_0} = \frac{(x-x_0)^k}{k!} \sum_{j=k}^{+\infty} \frac{q_j}{(j-k)!} (\rho_r(x-x_0))^{j-k}\Big|_{c=x_0} = \frac{(x-x_0)^k}{k!\rho_r^k} \left(\sum_{j=0}^{+\infty} \frac{q_j}{j!} (\rho_r(x-x_0))^j \right)_x^{(k)} \Bigg|_{c=x_0}$$
$$= \frac{(x-x_0)^k}{k!\rho_r^k} (f(\rho_r(x-x_0)))_x^{(k)}\Big|_{c=x_0}. \tag{50}$$

Therefore,

$$y = \sum_{j=0}^{+\infty} q_j \sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \mu_{rk} \binom{j}{k} \rho_r^{j-k} \frac{(x-x_0)^j}{j!}\Big|_{c=x_0} = \sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \mu_{rk} \sum_{j=0}^{+\infty} q_j \binom{j}{k} \rho_r^{j-k} \frac{(x-x_0)^j}{j!}\Big|_{c=x_0}$$
$$= \sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \mu_{rk} \frac{(x-x_0)^k}{k!\rho_r^k} (f(\rho_r(x-x_0)))_x^{(k)}\Big|_{c=x_0}. \tag{51}$$

Example 5. $y'' = 4y' - 4y; y = y(x; s_0, s_1)$; initial conditions $y(0; s_0, s_1) = s_0;\ (y(x; s_0, s_1))'_x|_{x=0} = s_1$.

The generalized differential operator for this differential equations is $D = s_1 D_{s_0} + 4(s_1 - s_0) D_{s_1}$.

Trivial transformations yield $Hr(D^j s;\ j = 0, 1, 2, \ldots) = 2$. Roots of the Hankel characteristic equation are $\rho_1 = \rho_2 = 2$. Then, $\mu_{10} = s_0$ and $\mu_{11} = s_1 - s_0$. Thus, $p_j = s_0 2^j + (s_1 - 2s_0)\binom{j}{1} 2^{j-1}$; and finally,

$$y(x; s_0, s_1) = s_0 \exp(2x) + (s_1 - 2s_0)\frac{x}{2}(\exp(2x))'_x = s_0 \exp(2x) + (s_1 - 2s_0)x\exp(2x).$$

This trivial example illustrates the above-mentioned fact that a solution cannot be expressed in a form comprising standard functions if roots of the Hankel characteristic equation are all not different.

4. The generalization of the Exp-function method

4.1. The algorithm for the construction of an exact solution of an ordinary differential equation

We will illustrate the algorithm for the initial problem of an nth order ordinary explicit differential equation:

$$\frac{d^n y}{dx^n} = P_n\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right);$$
$$\frac{d^j y}{dx^j}(x; s_0, s_1, \ldots, s_{n-1})\bigg|_{x=x_0} = s_j; x_0 \in R;\ P_n(c, s_0, s_1, \ldots, s_{n-1}) \in F_{x; s_1, \ldots, s_n}. \tag{52}$$

(A) The change of the independent variable $x = \varphi(z, a);\ z = \psi(x, a)$; where a is an independent parameter (in general, the variable change can comprise many independent parameters).
It can be noted that the selection of the invertible function $\varphi(z, a)$ is in many particular cases dependent on a subjective experience and knowledge of the area of application where the differential equation is used. The famous Exp-function method is based on the variable change $\varphi(z, a) := \exp(z)$. Then the differential equation takes the form defined by Eq. (34) and the image differential equation takes the form analogous to Eq. (35).

(B) Selection of the parameter b.
The sequence $(q_j;\ j \in Z_0)$ is constructed as follows: $q_0 := 1;\ q_{j+1} = \prod_{k=0}^{j}(a + bk);\ j = 0, 1, 2, \ldots$. It can be noted that the selection of the parameters a and b determines the standard function (Eq. (2)) which will figure in the expression of the exact solution (in most cases $a = 1$ is assumed).

(C) Construction of the generalized differential operator:

$$D := D_c + s_1 D_{s_0} + s_2 D_{s_1} + \cdots + s_{n-1} D_{s_{n-2}} + P_n(c, s_0, s_1, \ldots, s_{n-1}) D_{s_{n-1}}. \tag{53}$$

(D) Calculation of the first elements of the sequence $(\hat{p}_j;\ j \in Z_0)$ (Eq. (1)):

$$\hat{p}_j := \frac{1}{q_j} D^j s; \quad j = 0, 1, 2, \ldots, 2m. \tag{54}$$

(E) Construction of first elements of the Hankel matrixes sequence:

$$H_1 = [\hat{p}_0]; \quad H_2 = \begin{bmatrix} \hat{p}_0 & \hat{p}_1 \\ \hat{p}_1 & \hat{p}_2 \end{bmatrix}; \ldots; \quad H_{m+1} = \begin{bmatrix} \hat{p}_0 & \hat{p}_1 & \cdots & \hat{p}_m \\ \hat{p}_1 & \hat{p}_2 & \cdots & \hat{p}_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{p}_m & \hat{p}_{m+1} & \cdots & \hat{p}_{2m} \end{bmatrix}. \tag{55}$$

(F) Calculation of determinants of Hankel matrixes $d_k = \det H_k$ up to the first zero:

$$(d_1; d_2; \ldots; d_m; 0);\ d_1; d_2; \ldots; d_m \neq 0;\ d_{m+1} = 0. \tag{56}$$

(G) Construction of the mth Hankel's characteristic equation and determination of its roots $\rho_1; \rho_2; \ldots; \rho_m$:

$$\det \begin{bmatrix} \hat{p}_0 & \hat{p}_1 & \cdots & \hat{p}_m \\ \hat{p}_1 & \hat{p}_2 & \cdots & \hat{p}_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{p}_{m-1} & \hat{p}_m & \cdots & \hat{p}_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{bmatrix} = 0. \tag{57}$$

(H) If all roots of the characteristic equation are different, the following linear algebraic system of equations is solved for determination of $\mu_1; \mu_2; \ldots; \mu_m$:

$$\mu_1 \rho_1^j + \mu_2 \rho_2^j + \cdots + \mu_m \rho_m^j = \hat{p}_j; \quad j = 0, 1, \ldots, m-1. \tag{58}$$

(I) Following equalities must be tested:

$$D\mu_r = \mu_r \rho_r; D\rho_r = b\rho_r^2; \quad r = 1, 2, \ldots, m. \tag{59}$$

If these equalities hold, then statements (i), (ii) and (iii) of the Theorem 1 hold true too for the sequence $(D^j s;\ j = 0, 1, 2, \ldots)$. Moreover, $d_{m+1} = d_{m+2} = \cdots = 0$ and the exact solution of the differential equation can be expressed in a form comprising the selected standard function.

(J) Construction of the exact solution of the image differential equation in a form of a series:

$$\omega(z) = \sum_{j=0}^{+\infty} \frac{q_j}{j!} \left(\sum_{r=1}^{m} \mu_r \rho_r^j \right) (z - z_0)^j = \sum_{r=1}^{m} \mu_r \sum_{j=0}^{+\infty} \frac{q_j}{j!} \rho_r^j (z - z_0)^j. \tag{60}$$

(K) Expression of the exact solution in a form comprising the selected standard function $f(z) = \sum_{j=0}^{+\infty} \frac{q_j}{j!} z^j$:

$$\omega(z) = \sum_{r=1}^{m} \mu_r f(\rho_r (z - z_0)). \tag{61}$$

(L) Construction of the exact solution of the original differential equation:

$$y(x; s_1, \ldots, s_n) = \sum_{r=1}^{m} \mu_r f(\rho_r (\psi(x, a) - \psi(x_0, a))). \tag{62}$$

(M) Checking if the produced solution satisfies the original differential equation.

5. Examples

5.1. The exact solution is expressed in a ratio of sums of exponential functions

The initial problem reads:

$$y'_x = y^2 + y - 6;\ y = y(x; s);\ y(0; s) = s. \tag{63}$$

A variable change $x = \frac{1}{\alpha} \ln z;\ z = \exp(\alpha x)$ yields:

$$\omega'_z = \frac{1}{\alpha z} (\omega^2 + \omega - 6);\ \omega = \omega(z; s); \omega(1; s) = s. \tag{64}$$

We select $b = 1$. Thus, $q_j = j!;\ j \in Z_0$. The generalized differential operator reads:

$$D = D_c + \frac{1}{\alpha c} (s^2 + s - 6) D_s = D_c + \frac{(s+3)(s-2)}{\alpha c} D_s. \tag{65}$$

Computational symbolic transformations yield ($p_j := D^j s$):

$$\begin{aligned} &p_0 = s;\ p_1 = \frac{(s+3)(s-2)}{\alpha c};\ p_2 = \frac{(s+3)^2 (s-2)^2 (2s-1)}{\alpha^2 c^2} - \frac{(s+3)(s-2)}{\alpha c^2}, \\ &p_3 = \frac{(s+3)(s-2)(6s^2 + 6s - 11 - 6s\alpha - 3\alpha + 2\alpha^2)}{\alpha^3 c^3}, \ldots. \end{aligned} \tag{66}$$

Next, determinants of Hankel matrixes $d_n = \det(H_n);\ n = 1, 2, \ldots$ are calculated:

$$\begin{aligned} d_1 &= s;\ d_2 = \frac{(s+3)(s-2)(-s+sc-12)}{2\alpha^2c^2};\\ d_3 &= \frac{(s+3)^2(s-2)^2(\alpha-5)(\alpha+5)(2\alpha^2s-36\alpha+3s\alpha+25s)}{144\alpha^6c^6}. \end{aligned} \tag{67}$$

It can be noted, that equality $\alpha = 5$ produces $d_3 = 0$ at any s. Thus, $Hr\left(\frac{1}{j!}D^js\Big|_{\alpha=5}; j \in Z_0\right) = 2$. Therefore, the variable change $x = \frac{1}{5}\ln z; z = \exp(5x)$ appears to be appropriate.

Hankel characteristic equation reads:

$$\begin{vmatrix} s & Ds & \frac{1}{2}D^2s \\ Ds & \frac{1}{2}D^2s & \frac{1}{6}D^3s \\ 1 & \rho & \rho^2 \end{vmatrix} = 0. \tag{68}$$

Elementary transformations yield $\rho^2 - \frac{s-2}{5c}\rho = 0$. Roots are $\rho_1 = 0;\ \rho_2 = \frac{s-2}{5c}$. Thus, $\mu_1 = -3;\ \mu_2 = s+3$. We check the validity of the Theorem 1:

$$\begin{aligned} &D\mu_1 = \mu_1\rho_1 = 0;\ D\mu_2 = D(s+3) = \frac{(s+3)(s-2)}{5c} = \mu_2\rho_2;\\ &D\rho_1 = 0 = \rho_1^2;\ D\rho_2 = D\frac{s-2}{5c} = \left(\frac{s-2}{5c}\right)^2 = \rho_2^2. \end{aligned} \tag{69}$$

Thus, the sequence $(q_j;\ j \in Z_0)$ and the variable change were selected appropriately. Now,

$y = \sum_{j=0}^{+\infty} j!\left(-3\cdot 0^j + (s+3)\left(\frac{s-2}{5c}\right)^j\right)\Big|_{c=1}\frac{(z-1)^j}{j!} = -3 + (s+3)\sum_{j=0}^{+\infty}\left(\frac{(s-2)(z-1)}{5}\right)^j$, when $|z-1| < \frac{5}{|s-2|}$. Now we expand the produced series into the whole z-axis, except the point $z = \frac{s+3}{s-2}$. Then,

$$y = -3 + \frac{5s+15}{5-sz+z+2z-2} = \frac{2(3+s)-3(2-s)z}{3+s+(2-s)z}. \tag{70}$$

Finally, the exact solution of the differential equation reads:

$$y(x;s) = \frac{2(3+s)-3(2-s)\exp(5x)}{3+s+(2-s)\exp(5x)} = \frac{2(3+s)\exp(-3x)-3(2-s)\exp(2x)}{(3+s)\exp(-3x)+(2-s)\exp(2x)}. \tag{71}$$

5.2. The exact solution is expressed in a form comprising a square root

The initial problem reads:

$$y'_x = y^3; \quad y = y(x;s); y(c;s) = s. \tag{72}$$

Now we do not make a variable change, but select $b := 2$ and $q_0 = 1;\ q_j = (2j-1)!!;\ j = 1, 2, \ldots$ instead.

The generalized differential operator reads $D = s^3D_s$. Then, $p_0 = s;\ p_j = \frac{1}{q_j}D^js = s^{2j+1};\ j = 1, 2, \ldots$.

It is easy to see that the sequence of determinants of Hankel matrices is $(s; 0; 0; \ldots)$; thus $Hr(p_j;\ j \in Z_0) = 1$. The Hankel characteristic equation reads $\begin{vmatrix} s & s^3 \\ 1 & \rho \end{vmatrix} = 0$; or $\rho = s^2$. Then, $\mu = s$.

Now we need to check the validity of Theorem 1: $D\mu = s^3D_ss = s^3 = \mu\rho;\ D\rho = s^3D_ss^2 = 2s^4 = 2\rho^2$. Thus, the selection of the sequence $q_0 = 1;\ q_j = (2j-1)!!;\ j = 1, 2, \ldots$ is appropriate for this differential equation.

Now, the structure of the standard function yields:

$$y(x) = s + \sum_{j=1}^{+\infty}\frac{(2j-1)!!}{j!}s\cdot s^{2j}(x-c)^j = \frac{s}{\sqrt{1-2s^2(x-c)}}; |x-c| < \frac{1}{2s^2}. \tag{73}$$

Finally, the exact solution of the differential equation reads:

$$y(x;c) = \frac{s}{\sqrt{1-2s^2(x-c)}};\ x < \frac{1}{2s^2} + c \text{ if it is required that } y(x;c) \in R. \tag{74}$$

5.3. Partial solutions of the Liouville's equation

A natural question is if our developed technique could be applied for the construction of an exact analytical solution of an equation which is already solved by the classical Exp-function method. We select the Liouville's equation which is analyzed in [17]:

$$y \cdot y''_{xx} - (y'_x)^2 + 2y^3 = 0, \tag{75}$$

where the partial solution $y = y(x)$ satisfies initial conditions $y(0) = s$ and $y'_x(x)\big|_{x=0} = t$. We will seek only such solutions which can be expressed in a ratio of finite sums of exponential functions. We perform the variable change defined by Eq. (36) at $\alpha := 1$ because it helps to simplify technical computations. The image differential equation of Eq. (75) then takes the following form:

$$\omega(z^2\omega''_{zz} + z\omega'_z) - z^2(\omega'_z)^2 + 2\omega^3 = 0, \tag{76}$$

where $\omega = \omega(z);\ \omega(1) = s$ and $\omega'_z(z)\big|_{z=1} = t$. The generalized differential operation of the image differential equation reads:

$$D = D_c + tD_s + \frac{c^2t^2 - cst - 2s^3}{c^2 s} \cdot D_t. \tag{77}$$

Then, $\omega(z) = \sum_{j=0}^{+\infty} \frac{(z-1)^j}{j!} \cdot D^j s\Big|_{c=1}$ and $y(x) = \omega(\exp(x))$. We select $q_j := j!$ (Eq. (3)) and construct the sequence $(\hat{p}_j; j = 0, 1, 2, \ldots)$ where $\hat{p}_j = \frac{1}{j!} D^j s$.

The sequence of determinants of Hankel matrixes is:

$$\det H_1 = |\hat{p}_0| = s;$$

$$\det H_2 = \begin{vmatrix} \hat{p}_0 & \hat{p}_1 \\ \hat{p}_1 & \hat{p}_2 \end{vmatrix} = -\frac{c^2t^2 + sct + 2s^3}{2c^2};$$

$$\det H_3 = -\frac{c^2t^2 - s^2 + 4s^3}{144s^3c^6}(c^4t^4 + 3sc^3t^3 + 4c^2t^2s^3 + 2t^2s^2c^2 + 6ts^4c - 12s^6);$$

$$\det H_4 = \frac{(c^2t^2 - s^2 + 4s^3)^2}{1036800s^8c^{12}} \begin{pmatrix} -88t^2s^7c^2 - 72c^3t^3s^6 + 42c^4t^4s^5 + 48c^5t^5s^4 + 10c^6t^6s^3 + 1112t^2s^9c^2 + \\ +212s^6c^4t^4 + 360s^{11} + 236c^2t^2s^8 + 360ts^9c + 1440s^{12} + c^8t^8 + 6sc^7t^7 - \\ -18s^3c^5t^5 + 7s^2c^6t^6 - 44s^4c^4t^4 - 24s^5c^3t^3 + 1440cts^{10} + 456t^3s^7c^3 \end{pmatrix}.$$

Thus, $\det H_3|_{c=1} = \det H_4|_{c=1} = 0$ when

$$t^2 - s^2 + 4s^3 = 0, \tag{78}$$

or $t = \pm s\sqrt{1-4s};\ s \leqslant 1/4$. The graph of the curve (78) is illustrated in Fig. 1.

Thus, the solution of the differential Eq. (75) can be expressed in a ratio of finite sums of exponential functions when variables of initial conditions s and t satisfy equality (78). Now the algorithm described in Section 4.1 can be used to construct the function $\omega(z)$. It is sufficient to check if this function satisfies the original differential equation to conclude that it is a partial solution.

First of all, we construct the Hankel characteristic Eq. (57):

$$\det \begin{bmatrix} s & Ds & \frac{1}{2}D^2s \\ Ds & \frac{1}{2}D^2s & \frac{1}{6}D^3s \\ 1 & \rho & \rho^2 \end{bmatrix}\Bigg|_{\substack{c=1 \\ t=\pm s\sqrt{1-4s}}} = 0, \tag{79}$$

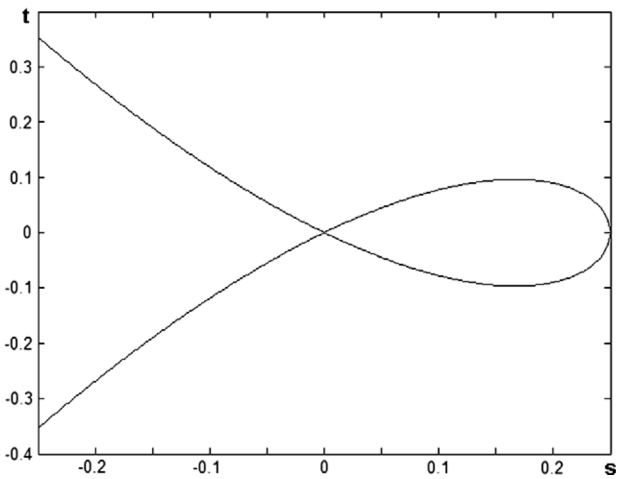

Fig. 1. The graph of the relationship between initial conditions s and t.

which yields $\rho_1 = \rho_2 = \frac{1}{2}\left(-1 \pm \sqrt{1-4s}\right)$. We assume $\rho = \rho_1 = \rho_2 = \frac{1}{2}\left(\sqrt{1-4s} - 1\right)$ and construct equalities:

$$\left.\frac{D^j s}{j!}\right|_{\substack{c=1\\ t=s\sqrt{1-4s}}} = \mu_1 \rho^j + \mu_2 \binom{j}{1} \rho^{j-1}; \quad j = 0,1,2,\ldots. \tag{80}$$

Eq. (80) yields a system of linear algebraic equations for determination of coefficients μ_1 and μ_2:

$$\left(\frac{1}{2}\left(\sqrt{1-4s}-1\right)\right)^j \mu_1 + \left(\frac{1}{2}\left(\sqrt{1-4s}-1\right)\right)^{j-1} \cdot j \cdot \mu_2 = \left.\frac{D^j s}{j!}\right|_{\substack{c=1\\ t=s\sqrt{1-4s}}}; \quad j = 0,1. \tag{81}$$

It can be noted that the assumption $\rho = \rho_1 = \rho_2 = \frac{1}{2}\left(-\sqrt{1-4s} - 1\right)$ would yield the same function $\omega = \omega(z)$.

The system of Eq. (81) yields $\mu_1 = s$ and $\mu_2 = \frac{s}{2}\left(1 + \sqrt{1-4s}\right)$. Finally,

$$\begin{aligned} \omega(z)|_{\substack{c=1\\ t=s\sqrt{1-4s}}} &= \sum_{j=0}^{+\infty} (z-1)^j \left(s\left(\frac{1}{2}\left(\sqrt{1-4s}-1\right)\right)^j + \frac{s}{2}\left(1+\sqrt{1-4s}\right) j \left(\frac{1}{2}\left(\sqrt{1-4s}-1\right)\right)^{j-1} \right) \\ &= \frac{2s\left(1-2s+\sqrt{1-4s}\right)z}{\left(2sz+1-2s+\sqrt{1-4s}\right)^2} = \frac{\frac{1-2s+\sqrt{1-4s}}{2s}\cdot z}{\left(z + \frac{1-2s+\sqrt{1-4s}}{2s}\right)^2} \end{aligned} \tag{82}$$

Further computations can be simplified introducing an auxiliary parameter $\beta := \frac{1-2s+\sqrt{1-4s}}{2s}$; $\beta \in R$. Now it is easy to check that the function $\bar{\omega}(z) := \omega(z)|_{\substack{c=1\\ t=s\sqrt{1-4s}}} = \frac{\beta \cdot z}{(\beta+z)^2}$ satisfies differential Eq. (76). Therefore,

$$y(x) = \bar{\omega}(\exp(x)) = \frac{\beta \exp(x)}{(\exp(x)+\beta)^2} \tag{83}$$

is a partial solutions of the differential Eq. (75) satisfying initial conditions $y(0) = \frac{\beta}{(\beta+1)^2}$ and $y'_x(x)|_{x=0} = \frac{\beta(\beta-1)}{(\beta+1)^3}$ for all β (except $\beta = -1$).

Thus, $Hr\left(\frac{1}{j!} D^j s\Big|_{\substack{c=1\\ t=\pm s\sqrt{1-4s}}}; \ j = 0,1,2,\ldots\right) = 2$ and equalities (80) hold true.

It can be noted, that Eq. (83) coincides with the solution presented in [17]. But we would like to stress that we did not guess the structure of the solution; it has been automatically identified by the direct application of the algorithm described in Section 4.1. This is the main difference between classical Exp-function type methods and our proposed technique.

But even more important fact is that our approach enabled to show that the solution of the differential Eq. (75) takes the form (83) only when initial conditions satisfy Eq. (78). This is an important fact, and we argue that the conclusion done in [17] is in general incorrect. The partial solution cannot be expressed in the form represented by Eq. (83) for any initial conditions. Our technique enabled the identification of the constraint linking two initial conditions. Classical Exp-function type methods are incapable of finding such constrains and may produce wrong results in general, what is clearly illustrated by this example.

6. Conclusions

We have constructed an analytical criterion determining if a solution of a nonlinear ordinary differential equation can be expressed in a form comprising standard functions. This criterion is much more general compared to the criterion presented in [12]. First of all, not only exponential functions are considered in the exact solution of a nonlinear differential equation. In fact, the new criterion works with any standard function. But the most important result is that this new criterion can be used to identify the structure of the solution which can be much more complex than a sum of standard functions or a ratio of sums of standard functions. That opens new possibilities for finding exact solutions of nonlinear differential equations.

New computational algorithm for automatic identification of the structure of the solution is constructed. Several examples are used to illustrate the proposed concept.

Recent developments of the Exp-function method were summarized in [18,19]; the Exp-function method has been used to solve differential-difference equations and stochastic equations [20–22]. Applicability of our technique for these problems is a definite object of future research.

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