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Comments on "A new algorithm for automatic computation of solitary wave solutions to nonlinear partial differential equations based on the Exp-function method"

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ABSTRACT

In their recent manuscript Lei Zhao et al. claim that they have constructed an algorithm for automatic construction of solitary wave solutions to nonlinear partial differential equations based on the Exp-function method. We argue that this algorithm does inherit all problems associated to the Exp-function method. We show that this algorithm produces solutions which do not satisfy the original differential equation. Even if the solutions would be correct, the Exp-function method would not be able to determine necessary and sufficient conditions for the existence of solitary solutions in the space of initial conditions and parameters.

1. Introduction

Lei Zhao et al. present an algorithm for automatic construction of solitary wave solutions to nonlinear partial differential equations based on the Exp-function method in [1]. Several examples are used to demonstrate the functionality of this algorithm. We argue that this algorithm does inherit all problems associated to the Exp-function method. We show that this algorithm produces solutions which do not satisfy the original differential equation. Moreover, we present an alternative analytical–computational framework for the construction of solitary solutions to nonlinear differential equations and point out the errors in [1].

2. Preliminaries

Let us consider the Equal-Width equation (Example 1, [1]):

$$\frac{\partial u}{\partial t} + 3u^2\frac{\partial u}{\partial x} - \alpha\frac{\partial^3 u}{\partial^2 x \partial t} = 0. \tag{1}$$

The change of variables $\lambda_1 t + \lambda_2 x = \eta$ transforms (1) into an ordinary nonlinear differential equation:

$$\lambda_1 u'_\eta(\eta) + 3\lambda_2 u^2(\eta)u'_\eta(\eta) - \alpha\lambda_1\lambda_2^2 u'''_{\eta\eta\eta}(\eta) = 0. \tag{2}$$

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Let us rewrite (2) in a more convenient general form:

$$w'''_{xxx} = 6\sigma^2(w^2 - w_0^2)w'_x; w = w(x, x_0, q, r, s), \tag{3}$$

where $\sigma,\ w_0,\ x_0 \in \mathrm{R}$; $w(x_0, x_0, q, r, s) = q$; $w'_x(x, x_0, q, r, s)\big|_{x=x_0} = r$; $w''_{xx}(x, x_0, q, r, s)\big|_{x=x_0} = s$. We will seek solitary solutions to (3) if only they do exist.

3. The extended Riccati equation

Let us consider the Riccati equation

$$y'_x = \gamma(y - y_1)(y - y_2); \quad y = y(x, x_0, q); \quad y(x_0, x_0, q) = q \tag{4}$$

where $\gamma \neq 0$; $\gamma,\ y_1,\ y_2 \in \mathbf{C}$. Note that the general form of the Riccati equation comprises $\gamma,\ y_1,\ y_2$ as functions of x [2], but we will consider only a simplified form when $\gamma,\ y_1,\ y_2$ are constant. The solution to (1) reads [2]:

$$y(x, x_0, q) = \frac{y_2(q - y_1)\exp(\gamma y_1(x - x_0)) - y_1(q - y_2)\exp(\gamma y_2(x - x_0))}{(q - y_1)\exp(\gamma y_1(x - x_0)) - (q - y_2)\exp(\gamma y_2(x - x_0))} \tag{5}$$

when $y_1 \neq y_2$, and

$$y(x, x_0, q) = y_0 + \frac{q - y_0}{1 - \gamma(q - y_0)(x - x_0)} \tag{6}$$

when $y_0 := y_1 = y_2$. (5) is the solitary solution and (6) is the degenerative solitary solution to (4).

The Riccati equation can be rearranged into the following forms [3]:

$$\text{(i)} \quad y'_x = a_2y^2 + a_1y + a_0; \tag{7}$$

$$\text{(ii)} \quad y''_{xx} = (2a_2y + a_1)y'_x = (2a_2y + a_1)(a_2y^2 + a_1y + a_0);$$
$$z''_{xx} = b_3z^3 + b_2z^2 + b_1z + b_0; \quad \text{(the Huxley equation [3])} \tag{8}$$

$$\text{(iii)} \quad z'''_{xxx} = (3b_3z^2 + 2b_2z + b_1)z'_x;$$
$$w'''_{xxx} = 6\sigma(w^2 - w_0^2)w'_x. \quad \text{(the Equal-Width equation [1] at } b_2 = 0) \tag{9}$$

Corollary 1. *All solutions to (4) are also solutions to (3). But solutions to (3) do satisfy (4) if and only if*

$$\begin{aligned} r &= a_2q^2 + a_1q + a_0 \\ s &= b_3q^3 + b_2q^2 + b_1q + b_0 \end{aligned} \tag{10}$$

In other words,

$$y(x, x_0, q) = w(x, x_0, q, (a_2q^2 + a_1q + a_0), (b_3q^3 + b_2q^2 + b_1q + b_0)). \tag{11}$$

The rigorous proof is given in [4].

3.1. The relationship between coefficients (γ, y_1, y_2) and (σ, w_0)

The Riccati equation (4) yields

$$z''_{xx} = 2\gamma^2\left(z - \frac{1}{2}(y_1 + y_2)\right)(z - y_1)(z - y_2) \tag{12}$$

which can be rearranged into the following form:

$$z''_{xx} = 2\gamma^2\left(z^3 - \frac{3}{2}(y_1 + y_2)z^2 + \frac{1}{2}((y_1 + y_2)^2 + 2y_1y_2)z - \frac{1}{2}(y_1 + y_2)y_1y_2\right). \tag{13}$$

Therefore,

$$w'''_{xxx} = 2\gamma^2(3w^2 - 3(y_1 + y_2)w + \frac{1}{2}((y_1 + y_2)^2 + 2y_1y_2))w'_x. \tag{14}$$

Note that the coefficient at w must be equal to 0. Therefore $y_1 = -y_2$. Let us denote

$$y_0^2 := y_1^2 = y_2^2. \tag{15}$$

Then, the Equal-Width equation takes the form:

$$w'''_{xxx} = 6\gamma^2\left(w^2 - \frac{y_0^2}{3}\right)w'_x. \tag{16}$$

Eqs. (3) and (16) do coincide. Therefore,

$$\sigma = \pm\gamma; \quad y_0^2 = 3w_0^2. \tag{17}$$

Corollary 2. *Solitary solution to the Equal-Width equation (3) is generated by the following Riccati equation:*

$$y'_x = \pm\sigma(y - \sqrt{3}w_0)(y + \sqrt{3}w_0). \tag{18}$$

3.2. The construction of solitary solutions

According to Corollary 2, solitary solutions to (3) read:

$$\begin{aligned} y(x, x_0, q) &= \frac{-\sqrt{3}w_0(q - \sqrt{3}w_0)\exp(\pm\sqrt{3}\sigma w_0(x - x_0)) - \sqrt{3}w_0(q + \sqrt{3}w_0)\exp(\mp\sqrt{3}\sigma w_0(x - x_0))}{(q - \sqrt{3}w_0)\exp(\pm\sqrt{3}\sigma w_0(x - x_0)) - (q + \sqrt{3}w_0)\exp(\mp\sqrt{3}\sigma w_0(x - x_0))} \\ &= w(x, x_0, q, \pm\sigma(q^2 - 3w_0^2), 2\sigma^2 q(q^2 - 3w_0^2)) \end{aligned} \tag{19}$$

Comment 1. The degenerate solitary solution is produced at $w_0 = 0$ ($y_0 = 0$):

$$y(x, x_0, q) = \frac{q}{1 \pm \sigma q(x - x_0)} = w(x, x_0, q, (\pm\sigma q^2), (2\sigma^2 q^3)). \tag{20}$$

Solitary solutions to the Equal-Width equation exist if and only if the initial conditions (q, r, s) do satisfy relationships

$$r = \pm\sigma(q^2 - 3w_0^2); \quad s = 2\sigma^2 q(q^2 - 3w_0^2). \tag{21}$$

Analogously, degenerate solitary solutions to the Equal-Width equation exist if and only if the initial conditions (s, t, v) do satisfy relationships $r = \pm\sigma q^2$ and $s = 2\sigma^2 q^3$.

Comment 2. The parameter σ (also the parameter γ) cannot be equal to zero because the Equal-Width equation does not exist then.

3.3. Computational experiments

Let us consider the Equal-Width equation at $\alpha = \frac{1}{6}$. Then (2) can be rearranged into the following form:

$$u'''_{\eta\eta\eta} = 6\left(\frac{3}{\lambda_1\lambda_2}u^2 + \frac{1}{\lambda_1^2}\right)u'_\eta. \tag{22}$$

Let $\eta = x + 3t$. Then, (22) yields:

$$u'''_{\eta\eta\eta} = 6(u^2 + 1)u'_\eta. \tag{23}$$

Then, according to (17), $\sigma = \pm 1$; $w_0^2 = -1$; $y_0^2 = -3$; $y_1 = \sqrt{3}i$; $y_2 = -\sqrt{3}i$. The Riccati equation (which does generate solitary solutions to (23)) reads:

$$y'_x = \pm(y^2 + 3). \tag{24}$$

The solitary solution to (24) reads:

$$y(x, x_0, q) = (-\sqrt{3}i)\frac{(q - \sqrt{3}i)\exp(\pm\sqrt{3}i(x - x_0)) + (q + \sqrt{3}i)\exp(\mp\sqrt{3}i(x - x_0))}{(q - \sqrt{3}i)\exp(\pm\sqrt{3}i(x - x_0)) - (q + \sqrt{3}i)\exp(\mp\sqrt{3}i(x - x_0))}. \tag{25}$$

Let us denote $\exp(\pm\sqrt{3}i(x - x_0)) := A + iB$. Then, $\exp(\mp\sqrt{3}i(x - x_0)) := A - iB$. Elementary transformations yield:

$$y(x, x_0, q) = \frac{\sqrt{3}qA + 3B}{\sqrt{3}A - qB}. \tag{26}$$

Thus, finally, the following pair of solitary solutions to the Equal-Width equation exist:

$$u_1(\eta,\eta_0,q,(q^2+3),2q(q^2+3)) = \frac{\sqrt{3}q\cos(\sqrt{3}(\eta-\eta_0)) + 3\sin(\sqrt{3}(\eta-\eta_0))}{\sqrt{3}\cos(\sqrt{3}(\eta-\eta_0)) - q\sin(\sqrt{3}(\eta-\eta_0))};$$
$$u_2(\eta,\eta_0,q,-(q^2+3),2q(q^2+3)) = \frac{\sqrt{3}q\cos(\sqrt{3}(\eta-\eta_0)) - 3\sin(\sqrt{3}(\eta-\eta_0))}{\sqrt{3}\cos(\sqrt{3}(\eta-\eta_0)) + q\sin(\sqrt{3}(\eta-\eta_0))}. \tag{27}$$

The graphical representations of u_1 and u_2 are shown in Fig. 1.

We will check the validity of results by performing the following computational experiment. We will construct the approximate numerical solution $\tilde{u}(\eta)$ to (23) at initial conditions $\eta_0 = 0$; $u_0 = u(\eta_0)$; $\dot{u}_0 = u'_\eta\big|_{\eta=\eta_0}$ and $\ddot{u}_0 = u''_{\eta\eta}\big|_{\eta=\eta_0}$ using constant step time forward marching techniques. Let us denote the approximate computational solution as $\tilde{u}(0+jh)$; $j = 0,1,2,\ldots$, where h is the step size. The exact analytical solution (27) is defined on the parameter lines $\dot{u}_0 = \pm(u_0^2+3)$ and $\ddot{u}_0 = 2u_0(u_0^2+3)$, but we release these constraints and assume that the solitary solution (27) is valid throughout the space of initial conditions. We travel 200 steps from the preselected tripartite of initial conditions and compute differences between the approximate numerical solution and the exact solution. Adding absolute differences for 200 steps produces an error estimate:

$$\varepsilon(u_0,\dot{u}_0,\ddot{u}_0) = \min_{k=1,2}\sum_{j=1}^{200}|\tilde{u}(0+jh) - u_k(0+jh,0,u_0,\dot{u}_0,\ddot{u}_0)|. \tag{28}$$

Unfortunately, a meaningful visualization of $\varepsilon(u_0,\dot{u}_0,\ddot{u}_0)$ in a 3D space poses a considerable complication. Note that constraints (21) define two separate curves in the space $(u_0,\dot{u}_0,\ddot{u}_0)$. In order to make the error plot comprehensible, we do perturb $\dot{u}_0$ and $\ddot{u}_0$ separately. In other words, we plot:

$$\varepsilon_1(u_0,\dot{u}_0,2u_0(u_0^2+3)) = \min_{k=1,2}\sum_{j=1}^{200}\left|\tilde{u}(0+jh) - u_k(0+jh,0,u_0,\dot{u}_0,2u_0(u_0^2+3))\right| \tag{29}$$

in Fig. 2. Numerical values of ε_1 higher than 8 are truncated to 8 in order to make the figure more comprehensive. It is clear that errors are almost equal to zero on the curves $\dot{u}_0 = \pm(u_0^2+3)$. Next, we compute

$$\varepsilon_{2,k}(u_0,(-1)^{k+1}(u_0^2+3),\ddot{u}_0) = \sum_{j=1}^{200}\left|\tilde{u}(0+jh) - u_k(0+jh,0,u_0,(-1)^{k+1}(u_0^2+3),\ddot{u}_0)\right|; k = 1,2 \tag{30}$$

and plot $\varepsilon_2 = \min_{k=1,2}\varepsilon_{2,k}$ in Fig. 3. Numerical values of ε_2 higher than 2 are truncated to 2; errors are almost equal to 0 on the curve $\ddot{u}_0 = 2u_0(u_0^2+3)$. Finally, we cut the phase space of initial conditions by the plane $u_0 = 0$ and compute

$$\varepsilon_3(0,\dot{u}_0,\ddot{u}_0) = \min_{k=1,2}\sum_{j=1}^{200}|\tilde{u}(0+jh) - u_k(0+jh,0,0,\dot{u}_0,\ddot{u}_0)|. \tag{31}$$

Numerical values of ε_3 higher than 2 are truncated to 2 (Fig. 4); errors are almost equal to 0 at points $(\dot{u}_0 = 3; \ddot{u}_0 = 0)$ and $(\dot{u}_0 = -3; \ddot{u}_0 = 0)$.

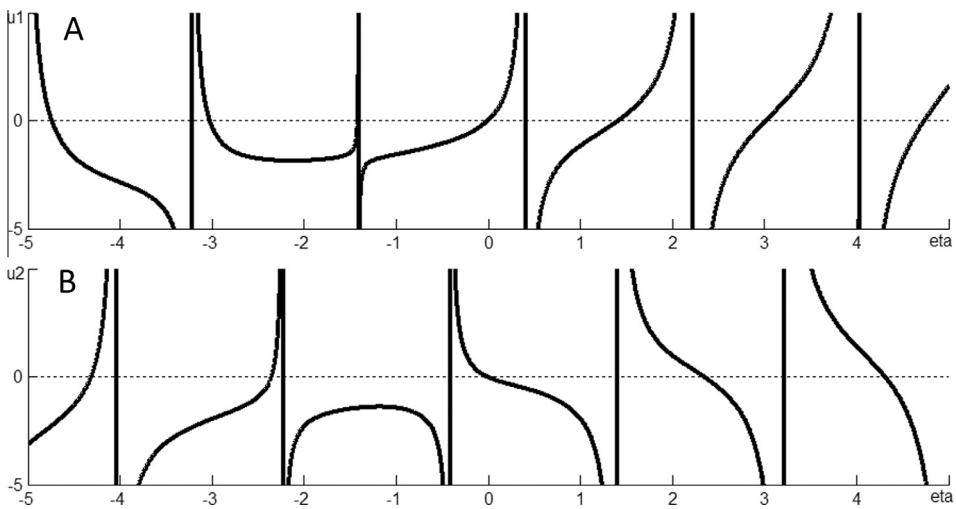

Fig. 1. Graphical representation of $u_1(\eta)$ (part A) and $u_2(\eta)$ (part B) at $\eta_0 = 0$. Note that thick solid vertical lines stand for infinite discontinuities.

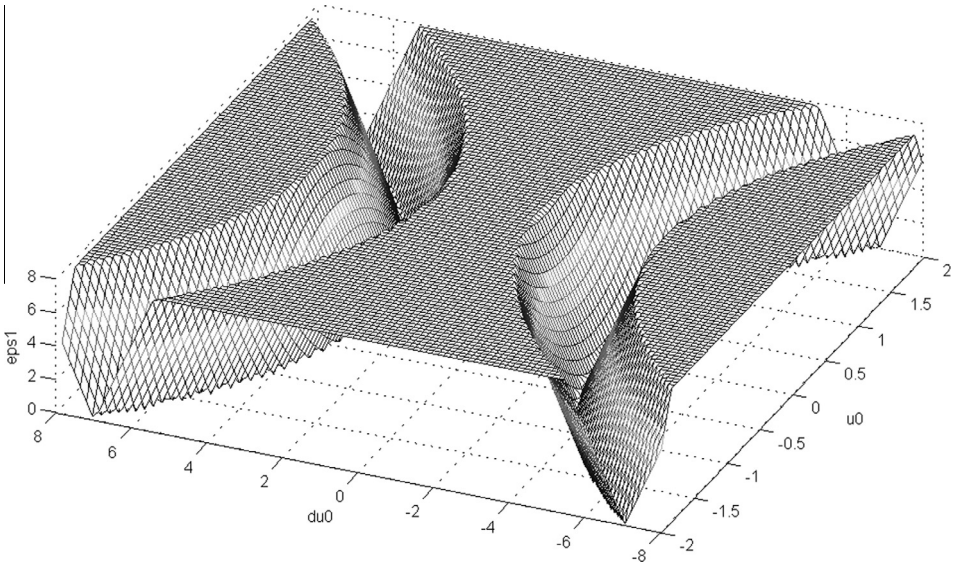

Fig. 2. The plot of $\varepsilon_1(u_0, \dot{u}_0, 2u_0(u_0^2+3))$. Errors are almost equal to 0 on the curves $\dot{u}_0 = \pm(u_0^2+3)$.

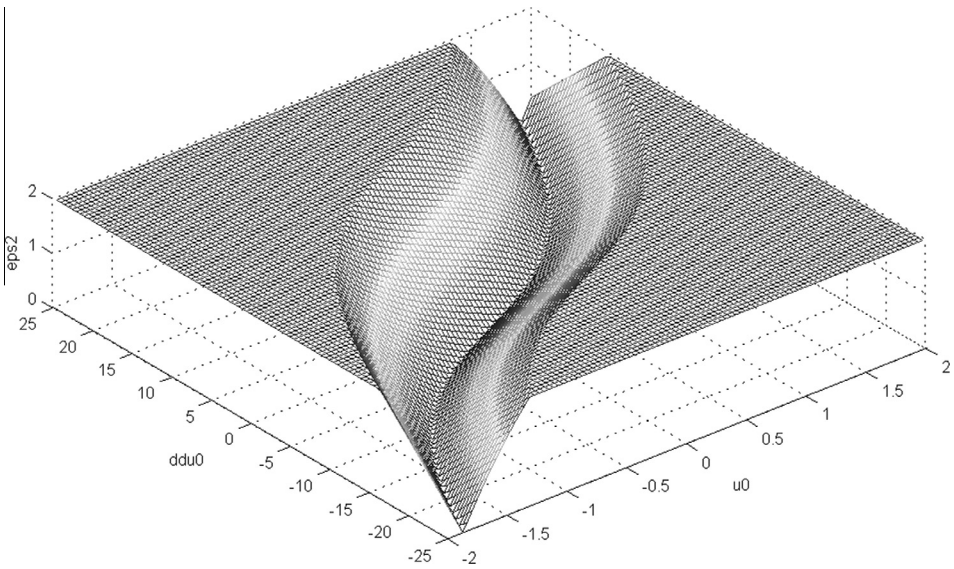

Fig. 3. Errors $\varepsilon_2 = \min_{k=1,2} \varepsilon_{2,k}$ are almost equal to 0 on the curve $\ddot{u}_0 = 2u_0(u_0^2+3)$.

4. Be careful with the Exp-function method

The solitary "solution" to (1) produced by the Exp-function method reads [1]:

$$u(x,t) = \frac{8a_0\alpha\lambda_1^2 b_1}{a_0^2\exp(-\lambda_1(\alpha t\lambda_1^2 + x)) + 8b_1^2\exp(\lambda_1(\alpha t\lambda_1^2 + x))\alpha\lambda_1^2}; \tag{32}$$

where a_0, b_1 are constants. It is clear that the structure of (32) is different from (19). But one can easily check that (32) does not satisfy (1); (32) is not a solution to (1).

We could finish our paper here, but we will make a step further. We will show where the hidden error is in [1]. In other words we will demonstrate what differential equation does generate the solution (32).

Let us consider a linear differential equation:

$$z''_{xx} - \gamma^2 z = 0. \tag{33}$$

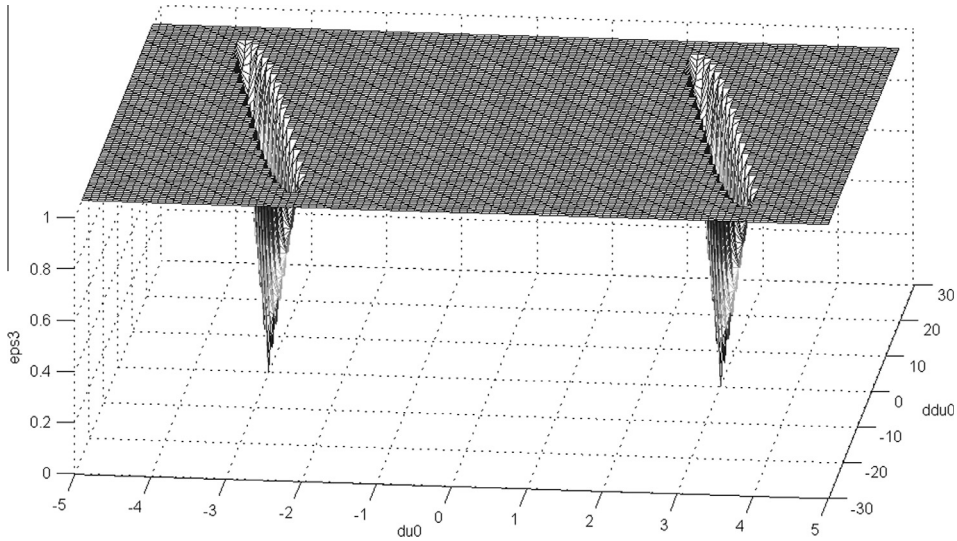

Fig. 4. Errors ε_3 are almost equal to 0 at points $(\dot{u}_0 = 3; \ddot{u}_0 = 0)$ and $(\dot{u}_0 = -3; \ddot{u}_0 = 0)$.

The solution to (33) reads:

$$z = \alpha \exp(\gamma(x - x_0)) + \beta \exp(-\gamma(x - x_0)); \alpha, \beta \in \mathbf{C}. \tag{34}$$

The change of variables $z = \frac{1}{y}$; $z'_x = -\frac{y'}{y^2}$; $z''_{xx} = \frac{2(y'_x)^2 - yy''_{xx}}{y^3}$ transforms (33) into the following form:

$$yy''_{xx} - 2(y'_x)^2 + \gamma y^2 = 0. \tag{35}$$

The solution to (35) reads:

$$y(x, x_0, \alpha, \beta) = \frac{1}{\alpha \exp(\gamma(x - x_0)) + \beta \exp(-\gamma(x - x_0))}. \tag{36}$$

Initial conditions $y(x_0, x_0, \alpha, \beta) = q$; $y'_x(x, x_0, \alpha, \beta)\big|_{x=x_0} = r$ yield equalities $\frac{1}{\alpha+\beta} = q$; $\frac{\gamma(\beta-\alpha)}{(\alpha+\beta)^2} = r$ what in its turn helps to derive the expressions of α and β:

$$\alpha = \frac{q\gamma - r}{2q^2\gamma}; \quad \beta = \frac{q\gamma + r}{2q^2\gamma}. \tag{37}$$

Thus, (36) reads:

$$y(x, x_0, q, r) = \frac{2q^2\gamma}{(q\gamma - r)\exp(\gamma(x - x_0)) + (q\gamma + r)\exp(-\gamma(x - x_0))}. \tag{38}$$

The differentiation of (35) in respect of x yields:

$$yy'''_{xxx} - 3y'_x y''_{xx} + 2\gamma y y'_x = 0, \tag{39}$$

which can be rearranged into the following form:

$$\begin{aligned} &w'''_{xxx} = 3\left(\frac{w''_{xx}}{w} - \frac{2}{3}\gamma\right)w'_x; \quad w = w(x, x_0, q, r, s); \\ &w(x_0, x_0, q, r, s) = q; \quad w'_x(x, x_0, q, r, s)\big|_{x=x_0} = r; \quad w''_x(x, x_0, q, r, s)\big|_{x=x_0} = s. \end{aligned} \tag{40}$$

The solution to (40) takes the form analogous to (32):

$$w(x, x_0, q, r, \frac{1}{q}(2r^2 - \gamma q^2)) = y(x, x_0, q, r). \tag{41}$$

Yet, it is crucial to note that (38) is a solution to (40) only on the following curve in the space of initial conditions:

$$s = \frac{1}{q}(2r^2 - \gamma q^2). \tag{42}$$

Another important observation is that (40) cannot be rearranged into the Equal-Width equation (3). Otherwise, the following equality should hold:

$$3\left(\frac{w''_{xx}}{w}-\frac{2}{3}\gamma\right)=6\sigma^2(w^2-w_0^2). \tag{43}$$

Note that (43) is the Huxley equation:

$$w''_{xx}=2\sigma^2w^3+\left(\frac{2}{3}\gamma-2\sigma^2w_0^2\right)w. \tag{44}$$

The solution to (44) can take the form of (5) or (6); but it is impossible for (38) to satisfy (44). That proves again, is an alternative way, that (32) is not a solution to (1).

5. Concluding remarks

The Exp-function method for the construction of solutions to nonlinear differential equations has attracted a huge amount of criticism during the last years. Seven typical errors done when using the Exp-function method are shown in [5]. Two additional typical errors done when using the Exp-function method are demonstrated in [6]. It is shown in [7] that many solitary solutions to nonlinear differential equations produced by the Exp-function method do not satisfy the original differential equation.

Lei Zhao et al. mention that their "solution" to the Equal-Width equation does coincide with the solution produced in [8]. We can just conclude that the automated system for the derivation of solitary solutions to nonlinear differential equations [1] inherits the worst features of the Exp-function method. It has been widely discussed and explicitly demonstrated that the Exp-function method is incapable to determine the conditions of the existence of solitary solutions in the space of initial conditions and system's parameters [9–12]. Now it appears that [1] generates wrong results also for the Equal-Width equation. The persistence of some authors to develop the infamous Exp-function method is surprising.

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