Existence of solitary solutions in systems of PDEs with multiplicative polynomial coupling

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A B S T R A C T

Necessary conditions for the existence of solitary solutions to systems of nonlinear partial differential equations with multiplicative polynomial coupling are derived in this paper. The inverse balancing technique is used to explicitly determine necessary existence conditions in terms of orders of the system and the nonlinearity. As the orders of the system increase, the order of solitary solutions does not increase monotonically. A computational framework for the derivation of additional constraints on the parameters of higher-order solitary solutions is presented.

1. Introduction

Increased capabilities of symbolic computations have sparked a resurgence of interest in mathematical techniques for the construction of analytical solutions to partial differential equations (PDEs). In particular, a large amount of attention has been directed towards the construction of solitary (also called soliton) solutions due to their significance in research fields ranging from physics and engineering to biology and medicine [1–3].

Solitary solutions are one of the primary tools used in the investigation of various real-world problems. Soliton behaviour in a reaction-diffusion system of PDEs applied for the simulation of myocardial beats is studied in [4]. Dynamics of solitary waves in water channels with sharp bends and branching points is considered in [5]. Models of superfluidity and superconductivity based on solitary waves are presented in [6,7]. It has recently been shown in [8] that tumour-induced angiogenesis can be controlled through solitons that are driving the process. Travelling solitary waves in nonlinear viscoelastic solids are discussed in [9].

Coupled systems of PDEs are considered in this paper. Typical examples of the analysis such systems in various fields of research are given below. A technique for despeckling ultrasound images based on coupled selective degenerate diffusion PDEs is presented in [10]. Steel hardering process is mathematically modeled using coupled PDEs describing temperature and carbon concentration in [11]. A parabolic-hyperbolic coupled system of Stokes-Lame PDEs governing fluid-structure interaction is considered in [12]. The Dirichlet problem for a class of non-linear coupled systems of reaction-diffusion PDEs is studied in [13].

A number of novel mathematical methods have been used to analyze solitary solutions to coupled PDEs in recent years. Soliton and soliton-like solutions of the coupled Schrödinger equations are studied using numerical and variational

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techniques in [14]. Orbital stability of solitary wave solutions to three-coupled long wave-short wave interaction equations is investigated in [15]. Multi-soliton solutions to the coupled Korteweg-de-Vries (KdV) equations are constructed in [16]. The Darboux transformation is used to construct soliton, breather and rogue wave solutions to the coupled Fokas-Lenells PDE system in [17]. The asymptotic stability of solitary wave solutions to the Schrödinger equation coupled with nonlinear oscillators is investigated in [18].

The main objective of this paper is to derive necessary existence conditions for solitary solutions to the following system of PDEs coupled with multiplicative polynomial terms:

$$\sum_{j=1}^{n}\sum_{s=0}^{j} a_{s,j-s}\frac{\partial^j \mathcal{U}}{\partial x^s \partial t^{j-s}} = \sum_{j=0}^{k}\sum_{s=0}^{j} b_{j-s,s}\mathcal{U}^{j-s}\mathcal{V}^s; \tag{1}$$

$$\sum_{j=1}^{m}\sum_{s=0}^{j} c_{s,j-s}\frac{\partial^j \mathcal{V}}{\partial x^s \partial t^{j-s}} = \sum_{j=0}^{l}\sum_{s=0}^{j} d_{j-s,s}\mathcal{U}^{j-s}\mathcal{V}^s, \tag{2}$$

where $\mathcal{U} = \mathcal{U}(t,x)$, $\mathcal{V} = \mathcal{V}(t,x)$; $n, m, k, l \in \mathbb{N}$; coefficients $a_{s,j-s}, b_{j-s,s}, c_{s,j-s}, d_{j-s,s} \in \mathbb{R}$.

The standard independent variable substitution $z := \alpha x + \beta t$; $\alpha, \beta \neq 0$ transforms (1), (2) into a system of ordinary differential equations (ODEs) with multiplicative polynomial coupling:

$$u_z^{(n)} + a_{n-1}u_z^{(n-1)} + \cdots + a_1 u_z' = \sum_{j=0}^{k} Q_j^{(b)}(u,v); \tag{3}$$

$$v_z^{(m)} + c_{m-1}v_z^{(m-1)} + \cdots + c_1 v_z' = \sum_{j=0}^{l} Q_j^{(d)}(u,v), \tag{4}$$

where $u = u(z) = \mathcal{U}(t,x)$; $v = v(z) = \mathcal{V}(t,x)$; coefficients a_j, $j = 1,\ldots,n-1$; c_j, $j = 1,\ldots,m-1$ are linear combinations of $a_{s,j-s}$ and $c_{s,j-s}$ respectively. Functions $Q_j^{(b)}, Q_j^{(d)}$ read:

$$Q_j^{(b)}(u,v) := \sum_{s=0}^{j} b_{j-s,s}u^{j-s}v^s; \tag{5}$$

$$Q_j^{(d)}(u,v) := \sum_{s=0}^{j} d_{j-s,s}u^{j-s}v^s. \tag{6}$$

Solitary solutions of the following form are considered:

$$u_0 = \sigma\frac{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right) - u_j\right)}{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right) - z_j\right)}; \tag{7}$$

$$v_0 = \gamma\frac{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right) - v_j\right)}{\prod_{j=1}^{N}\left(\exp\left(\eta(z-z_0)\right) - z_j\right)}, \tag{8}$$

where $N \in \mathbb{N}$; $\sigma, \gamma, \eta \in \mathbb{R}$; $u_j, v_j, z_j \in \mathbb{C}$; $u_s \neq z_j$; $v_s \neq z_j$; $s, j = 1,\ldots,N$.

The inverse balancing technique is used to determine necessary existence conditions of solitary solutions in terms of the system's derivative orders (n, m), nonlinearity orders (k, l) and solitary solution order N. The underlying idea of this technique is the assumption that the solution parameters are known and the system parameters can be expressed in terms of the solution parameters. Since systems (3), (4) is linear in coefficients a_j, $b_{j,s}$, c_j, $d_{j,s}$, the inverse balancing procedure results in a system of linear algebraic equations with respect to a_j, $b_{j,s}$, c_j, $d_{j,s}$. The necessary conditions for the existence of (7), (8) in (3), (4) are equivalent to non-degeneracy conditions of the obtained algebraic linear system.

2. Derivation of necessary existence conditions for solitary solutions

2.1. Transformation of the system and solitary solutions

The standard independent variable transformation is used to simplify the analysis of (3), (4) and (7), (8):

$$\widehat{z} := \exp\left(\eta(z - z_0)\right). \tag{9}$$

Using (9), the solitary solution (7), (8) reads:

$$u_0 = \widehat{u}_0(\widehat{z}) = \sigma \frac{U(\widehat{z})}{Z(\widehat{z})}; \quad v_0 = \widehat{v}_0(\widehat{z}) = \gamma \frac{V(\widehat{z})}{Z(\widehat{z})}, \tag{10}$$

where

$$Z(\widehat{z}) := \prod_{j=1}^{N} \left(\widehat{z} - z_j\right); \quad U(\widehat{z}) := \prod_{j=1}^{N} \left(\widehat{z} - u_j\right); \quad V(\widehat{z}) := \prod_{j=1}^{N} \left(\widehat{z} - v_j\right). \tag{11}$$

Note that

$$Z\left(z_j\right) = U\left(u_j\right) = V\left(v_j\right) = 0; \quad j = 1, \dots, N. \tag{12}$$

Let $\widehat{u}(\widehat{z}) = u(z)$ and $\widehat{v}(\widehat{z}) = v(z)$. Then:

$$u'_z = \eta \widehat{z} \widehat{u}'_{\widehat{z}}; \quad u_z^{(j)} = \left(\eta \widehat{z} \widehat{u}'_{\widehat{z}}\right)_{\widehat{z}}^{(j-1)} = \eta^j \sum_{s=1}^{j} f_{js} \widehat{z}^s \widehat{u}_{\widehat{z}}^{(s)}; \quad j = 2, 3, \dots; \quad f_{js} \in \mathbb{R}. \tag{13}$$

Analogous relations hold for derivatives of $v(z)$:

$$v'_z = \eta \widehat{z} \widehat{v}'_{\widehat{z}}; \quad v_z^{(j)} = \left(\eta \widehat{z} \widehat{v}'_{\widehat{z}}\right)_{\widehat{z}}^{(j-1)} = \eta^j \sum_{s=1}^{j} g_{js} \widehat{z}^s \widehat{v}_{\widehat{z}}^{(s)}; \quad j = 2, 3, \dots; \quad g_{js} \in \mathbb{R}. \tag{14}$$

Inserting (13), (14) into (3), (4) yields:

$$\eta^n \widehat{z}^n \widehat{u}_{\widehat{z}}^{(n)} + A_{n-1} \widehat{z}^{n-1} \widehat{u}_{\widehat{z}}^{(n-1)} + \dots + A_1 \eta \widehat{z} \widehat{u}'_{\widehat{z}} = \sum_{j=0}^{k} Q_j^{(b)}(\widehat{u}, \widehat{v}); \tag{15}$$

$$\eta^m \widehat{z}^m \widehat{v}_{\widehat{z}}^{(m)} + C_{m-1} \widehat{z}^{m-1} \widehat{v}_{\widehat{z}}^{(m-1)} + \dots + C_1 \eta \widehat{z} \widehat{v}'_{\widehat{z}} = \sum_{j=0}^{l} Q_j^{(d)}(\widehat{u}, \widehat{v}), \tag{16}$$

where coefficients A_j; $j = 1, \dots, n-1$ and C_s; $s = 1, \dots, m-1$ are linear combinations of η^j, f_{js}, a_j and η^j, g_{js}, c_j respectively.

2.2. Balancing the orders of derivative and nonlinear terms

Inserting solitary solutions $\widehat{u}_0, \widehat{v}_0$ into $Q^{(b)}$ results in:

$$Q_j^{(b)}(\widehat{u}_0, \widehat{v}_0) = \sum_{s=0}^{j} b_{j-s,s} \frac{U^{j-s}(\widehat{z}) V^s(\widehat{z})}{Z^{j-s}(\widehat{z}) Z^s(\widehat{z})} = \frac{1}{Z^j} \sum_{s=0}^{j} b_{j-s,s} U^{j-s} V^s. \tag{17}$$

Analogously,

$$Q_j^{(d)}(\widehat{u}_0, \widehat{v}_0) = \frac{1}{Z^j} \sum_{s=0}^{j} d_{j-s,s} U^{j-s} V^s. \tag{18}$$

Derivatives of $\widehat{u}_0, \widehat{v}_0$ read:

$$(\widehat{u}_0)_{\widehat{z}}^{(j)} = \frac{\widetilde{U}_{N,j}}{Z^{j+1}}; \quad (\widehat{v}_0)_{\widehat{z}}^{(j)} = \frac{\widetilde{V}_{N,j}}{Z^{j+1}}, \tag{19}$$

where $\widetilde{U}_{N,j}, \widetilde{V}_{N,j}$ are order $(j+1)(N-1)$ polynomials in $\widehat{z}$.

Inserting (17)–(19) into (15), (16) yields:

$$\eta^n \widehat{z}^n \frac{\widetilde{U}_{N,n}}{Z^{n+1}} + A_{n-1} \widehat{z}^{n-1} \frac{\widetilde{U}_{N,n-1}}{Z^n} + \dots + A_1 \widehat{z} \frac{\widetilde{U}_{N,1}}{Z^2} = \sum_{j=0}^{k} \left(\frac{1}{Z^j} \sum_{s=0}^{j} b_{j-s,s} U^{j-s} V^s \right); \tag{20}$$

$$\eta^m \widehat{z}^m \frac{\widetilde{V}_{N,m}}{Z^{m+1}} + C_{m-1}\widehat{z}^{n-1}\frac{\widetilde{V}_{N,m-1}}{Z^m} + \cdots + C_1 \widehat{z}\frac{\widetilde{V}_{N,1}}{Z^2} = \sum_{j=0}^{l}\left(\frac{1}{Z^j}\sum_{s=0}^{j} d_{j-s,s}U^{j-s}V^s\right). \tag{21}$$

Equality (20) can hold true only if the orders of denominator polynomials are balanced:

$$k = n + 1. \tag{22}$$

Analogously, (21) must satisfy:

$$l = m + 1. \tag{23}$$

Conditions (22), (23) have been previously noted in literature for single ODEs [19–21].

2.3. Inverse balancing of system coefficients

The inverse balancing technique is based on the determination of system coefficients in terms of solution parameters using polynomials (20), (21). Let balancing conditions (22), (23) hold true. Multiplying (20) and (21) by $\frac{1}{\eta^n}Z^{n+1}$ and $\frac{1}{\eta^m}Z^{m+1}$ respectively results in:

$$\begin{aligned} &\widehat{z}^n\widetilde{U}_{N,n} + \frac{A_{n-1}}{\eta^n}\widehat{z}^{n-1}\widetilde{U}_{N,n-1}Z + \cdots + \frac{A_1}{\eta^n}\widehat{z}\widetilde{U}_{N,1}Z^{n-1} \\ &= \frac{1}{\eta^n}\sum_{j=0}^{n+1}\left(Z^{n-j+1}\sum_{s=0}^{j} b_{j-s,s}U^{j-s}V^s\right); \end{aligned} \tag{24}$$

$$\begin{aligned} &\widehat{z}^m\widetilde{V}_{N,m} + \frac{C_{m-1}}{\eta^m}\widehat{z}^{m-1}\widetilde{V}_{N,m-1}Z + \cdots + \frac{C_1}{\eta^m}\widehat{z}\widetilde{V}_{N,1}Z^{m-1} \\ &= \frac{1}{\eta^m}\sum_{j=0}^{m+1}\left(Z^{m-j+1}\sum_{s=0}^{j} d_{j-s,s}U^{j-s}V^s\right). \end{aligned} \tag{25}$$

Gathering coefficients of like powers of $\widehat{z}$ and setting them to zero results in a system of linear algebraic equations. Eq. (24) yields $q_{N,n}^{(1)} := (n+1)N+1$ linear equations with respect to $p_n^{(1)} := \frac{1}{2}(n+2)^2 + \frac{3}{2}n$ unknown coefficients $A_1,\dots,A_{n-1}; b_{j-s,s}; j = 0,\dots,n+1; s = 0,\dots,j$. Analogously, (25) yields $q_{N,m}^{(2)} := (m+1)N+1$ linear equations with respect to $p_m^{(2)} := \frac{1}{2}(m+2)^2 + \frac{3}{2}m$ unknowns $C_1,\dots,C_{m-1}; d_{j-s,s}; j = 0,\dots,m+1; s = 0,\dots,j$.

Thus, the dimension of the linear system obtained from (24), (25) reads:

$$q_{N,n,m} := q_{N,n}^{(1)} + q_{N,m}^{(2)} = (n+m+2)N + 2. \tag{26}$$

The obtained linear system has the following number of unknowns:

$$p_{n,m} := p_n^{(1)} + p_m^{(2)} = \frac{1}{2}\left((n+2)^2 + (m+2)^2 + 3(n+m)\right); \tag{27}$$

and depends on the following number of solitary solution parameters:

$$r_N := 3N + 3. \tag{28}$$

The obtained system can only be solved if the number of equations does not exceed the total number of unknowns and parameters of solitary solutions:

$$q_{N,n,m} \le p_{n,m} + r_N. \tag{29}$$

Note that parameters of the solitary solutions can be used to reduce the rank of the linear system, making it solvable. Inequality (29) can be simplified as:

$$2(n+m-1)N - 2 \le (n+2)^2 + (m+2)^2 + 3(n+m). \tag{30}$$

Condition (30) is the necessary condition for the existence of solitary solutions (7), (8) to (3), (4) (and, by extension, to (1), (2)). Inequality (30) is illustrated in Fig. 1.

Note that if the condition

$$q_{N,n,m} \le p_{n,m}; \tag{31}$$

holds true, solitary solutions can exist without imposing any constraints on parameters $\eta; \sigma; \gamma; z_1,\dots,z_N; u_1,\dots,u_N; v_1,\dots,v_N$. If (31) does not hold true – but (30) is satisfied – the parameters of the solitary solution can be used to reduce the rank of the obtained linear system.

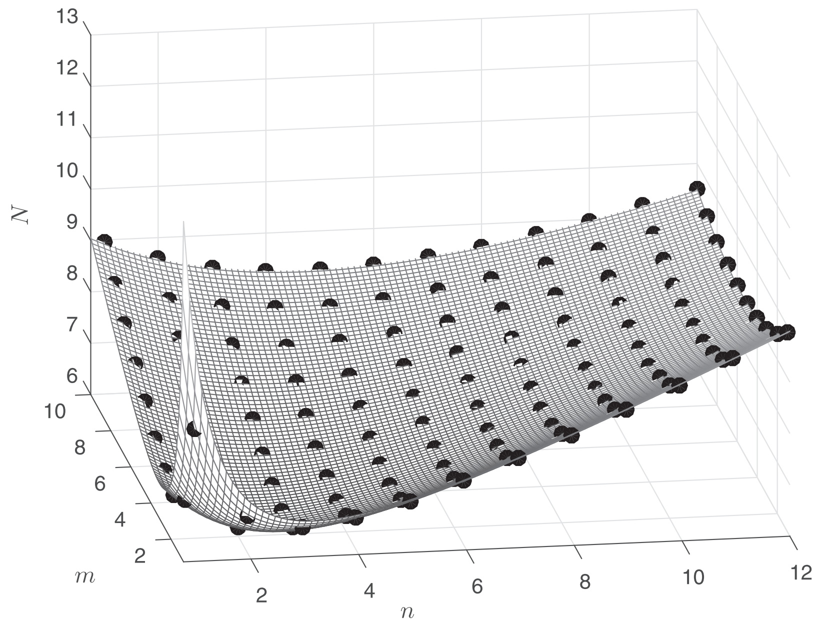

Fig. 1. Plot illustrating condition (30). For given orders n, m of the considered differential equations, the necessary existence condition of solitary solutions can only be satisfied if the solution order N does not lie above the given surface. Black circles denote maximum values of N for integer values of n, m.

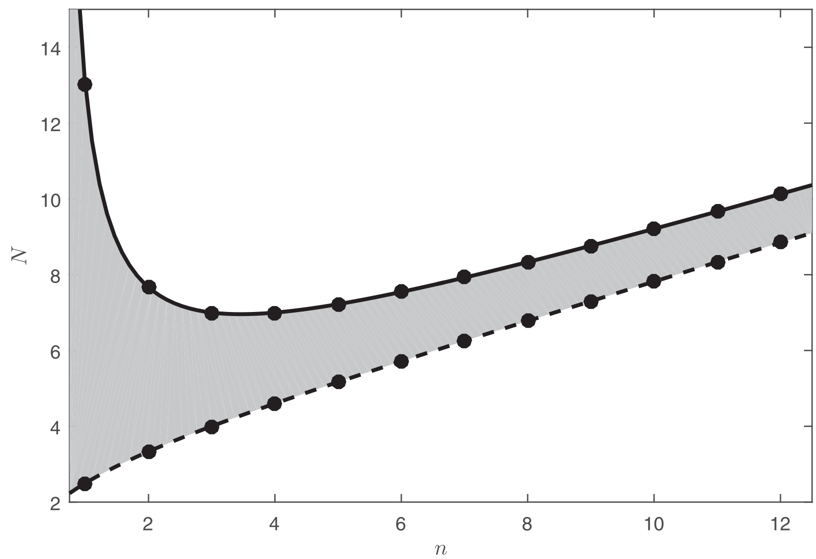

Fig. 2. Plot illustrating conditions (32) and (33). The necessary existence condition of solitary solutions is only satisfied if N does not lie above the solid black curve. Furthermore, if N lies below the dashed line, no additional constraints on the parameters of the solitary solutions need to be imposed. Solitary solutions can only exist with additional constraints on their parameters for values of N lying inside the shaded region. Black circles indicate integer values of equation order n which are used to construct Table 1.

If the orders of equations ((3), (4)) coincide $(n = m)$, conditions (30) and (31) can be simplified as:

$$(2n-1)N - 1 \le (n+2)^2 + 3n; \tag{32}$$

and

$$(2n+2)N + 2 \le (n+2)^2 + 3n. \tag{33}$$

Fig. 2 and Table 1 display cases of n, N for which necessary existence conditions are satisfied.

3. Example: the first order system

Let us consider the case $n = m = 1$. System (3), (4) reads:

$$u'_z = b_{00} + b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2; \quad (34)$$

$$v'_z = d_{00} + d_{10}u + d_{01}v + d_{20}u^2 + d_{11}uv + d_{02}v^2. \quad (35)$$

By (32), the above system satisfies necessary existence conditions of solitary solutions of order up to $N = 13$. Second order solitary solutions in the case $b_{01} = d_{10} = 0$ have been investigated in [22]. Cases $N = 1, 2, 3$ are considered below. We will demonstrate that no constraints on solitary solution parameters need to be imposed for $N = 1, 2$. Also, we will show that solitary solutions of order $N = 3$ can only exist with constraints on their parameters.

Substitution (9) transforms (34), (35) into:

$$\eta\widehat{z}\widehat{u}'_{\widehat{z}} = b_{00} + b_{10}\widehat{u} + b_{01}\widehat{v} + b_{20}\widehat{u}^2 + b_{11}\widehat{u}\widehat{v} + b_{02}\widehat{v}^2; \quad (36)$$

$$\eta\widehat{z}\widehat{v}'_{\widehat{z}} = d_{00} + d_{10}\widehat{u} + d_{01}\widehat{v} + d_{20}\widehat{u}^2 + d_{11}\widehat{u}\widehat{v} + d_{02}\widehat{v}^2. \quad (37)$$

Equations (24), (25) yield:

$$\eta\sigma\widehat{z}\left(ZU'_{\widehat{z}} - UZ'_{\widehat{z}}\right) = b_{00}Z^2 + b_{10}\sigma UZ + b_{01}\gamma VZ + b_{20}\sigma^2U^2 + b_{11}\sigma\gamma UV + b_{02}\gamma^2V^2; \quad (38)$$

$$\eta\gamma\widehat{z}\left(ZV'_{\widehat{z}} - VZ'_{\widehat{z}}\right) = d_{00}Z^2 + d_{10}\sigma UZ + d_{01}\gamma VZ + d_{20}\sigma^2U^2 + d_{11}\sigma\gamma UV + d_{02}\gamma^2V^2. \quad (39)$$

3.1. Kink solitary solutions – case $N = 1$

Solitary solutions (10) are called kink solutions for $N = 1$ [23]. In this case $Z(\widehat{z}) = \widehat{z} - z_1; U(\widehat{z}) = \widehat{z} - u_1; V(\widehat{z}) = \widehat{z} - v_1$. Note that the fact that (38), (39) are quadratic polynomials results in six linear equations with respect to coefficients b_{js}, d_{js}. Taking $\widehat{z} = z_1$ in (38), (39) yields two linear equations:

$$-\eta\sigma z_1U(z_1) = b_{20}\sigma^2U^2(z_1) + b_{11}\sigma\gamma U(z_1)V(z_1) + b_{02}\gamma^2V^2(z_1); \quad (40)$$

$$-\eta\gamma z_1V(z_1) = d_{20}\sigma^2U^2(z_1) + d_{11}\sigma\gamma U(z_1)V(z_1) + d_{02}\gamma^2V^2(z_1). \quad (41)$$

Another pair of equations is obtained by taking $\widehat{z} = u_1$, $\widehat{z} = v_1$ in (38) and (39) respectively:

$$\eta\sigma u_1Z(u_1) = b_{00}Z^2(u_1) + b_{01}\gamma V(u_1)Z(u_1) + b_{02}\gamma^2V^2(u_1); \quad (42)$$

$$\eta\gamma v_1Z(v_1) = d_{00}Z^2(v_1) + d_{10}\sigma U(v_1)Z(v_1) + d_{20}\sigma^2U^2(v_1). \quad (43)$$

Finally, $\widehat{z} = 0$ results in the following pair of linear equations:

$$b_{00}z_1^2 + b_{10}\sigma u_1z_1 + b_{01}\gamma v_1z_1 + b_{20}\sigma^2u_1^2 + b_{11}\sigma\gamma u_1v_1 + b_{02}\gamma^2v_1^2 = 0; \quad (44)$$

$$d_{00}z_1^2 + d_{10}\sigma u_1z_1 + d_{01}\gamma v_1z_1 + d_{20}\sigma^2u_1^2 + d_{11}\sigma\gamma u_1v_1 + d_{02}\gamma^2v_1^2 = 0. \quad (45)$$

Note that there are 6 linear equations with 12 unknowns, thus some coefficients b_{js}, d_{js} can be chosen freely, while others are obtained from (40)–(45). Solving (40), (42) for b_{01}, b_{11} yields:

$$b_{01} = -\frac{b_{02}\gamma^2V^2(u_1) - \eta\sigma u_1Z(u_1) + b_{00}Z^2(u_1)}{\gamma V(u_1)Z(u1)}; \quad (46)$$

$$b_{11} = -\frac{b_{02}\gamma^2V^2(z_1) + b_{20}\sigma^2U^2(z_1) + \eta\sigma z_1U(z_1)}{\sigma\gamma U(z_1)V(z_1)}. \quad (47)$$

Analogously, (41), (43) yield:

$$d_{10} = -\frac{d_{20}\sigma^2U^2(v_1) - \eta\gamma v_1Z(v_1) + d_{00}Z^2(v_1)}{\sigma U(v_1)Z(v_1)}; \quad (48)$$

$$d_{11} = -\frac{d_{02}\gamma^2V^2(z_1) + d_{20}\sigma^2U^2(z_1) + \eta\gamma z_1V(z_1)}{\sigma\gamma U(z_1)V(z_1)}. \quad (49)$$

Two of the remaining coefficients b_{js}, d_{js} can be obtained directly from (44) and (45). The six remaining coefficients can be chosen arbitrarily.

The described computations prove that systems of the form (34), (35) do admit kink solitary solutions. Note that no constraints on the parameters of the solution had to be imposed, because condition (33) holds true.

3.2. Bright/dark solitary solutions – case $N = 2$

Taking $N = 2$ in (10) results in bright/dark solitary solutions [23]. Functions Z, U, V read:

$$Z(\widehat{z}) = (\widehat{z} - z_1)(\widehat{z} - z_2); \quad U(\widehat{z}) = (\widehat{z} - u_1)(\widehat{z} - u_2); \quad V(\widehat{z}) = (\widehat{z} - v_1)(\widehat{z} - v_2). \tag{50}$$

Equations (38), (39) describe quartic polynomials in $\widehat{z}$, which yield 10 linear equations.
Taking $\widehat{z} = z_1, z_2$ in (38) results in:

$$\eta\sigma z_1(z_2 - z_1)U(z_1) = b_{20}\sigma^2U^2(z_1) + b_{11}\sigma\gamma U(z_1)V(z_1) + b_{02}\gamma^2V^2(z_1); \tag{51}$$

$$\eta\sigma z_2(z_1 - z_2)U(z_2) = b_{20}\sigma^2U^2(z_2) + b_{11}\sigma\gamma U(z_2)V(z_2) + b_{02}\gamma^2V^2(z_2). \tag{52}$$

Analogous computations with respect to (39) result in:

$$\eta\gamma z_1(z_2 - z_1)V(z_1) = d_{20}\sigma^2U^2(z_1) + d_{11}\sigma\gamma U(z_1)V(z_1) + d_{02}\gamma^2V^2(z_1); \tag{53}$$

$$\eta\gamma z_2(z_1 - z_2)V(z_2) = d_{20}\sigma^2U^2(z_2) + d_{11}\sigma\gamma U(z_2)V(z_2) + d_{02}\gamma^2V^2(z_2). \tag{54}$$

Taking $\widehat{z} = u_1, u_2$ in (38) and $z = v_1, v_2$ in (39) yield:

$$\eta\sigma u_1(u_1 - u_2)Z(u_1) = b_{00}Z^2(u_1) + b_{01}\gamma V(u_1)Z(u_1) + b_{02}\gamma^2V^2(u_1); \tag{55}$$

$$\eta\sigma u_2(u_2 - u_1)Z(u_2) = b_{00}Z^2(u_2) + b_{01}\gamma V(u_2)Z(u_2) + b_{02}\gamma^2V^2(u_2), \tag{56}$$

and

$$\eta\gamma v_1(v_1 - v_2)Z(v_1) = d_{00}Z^2(v_1) + d_{10}\sigma U(v_1)Z(v_1) + d_{20}\sigma^2U^2(v_1); \tag{57}$$

$$\eta\gamma v_2(v_2 - v_1)Z(v_2) = d_{00}Z^2(v_2) + d_{10}\sigma U(v_2)Z(v_2) + d_{20}\sigma^2U^2(v_2). \tag{58}$$

Finally, $\widehat{z} = 0$ results in the following pair of linear equations:

$$b_{00}z_1^2z_2^2 + b_{10}\sigma u_1u_2z_1z_2 + b_{01}\gamma v_1v_2z_1z_2 + b_{20}\sigma^2u_1^2u_2^2 + b_{11}\sigma\gamma u_1u_2v_1v_2 + b_{02}\gamma^2v_1^2v_2^2 = 0; \tag{59}$$

$$d_{00}z_1^2z_2^2 + d_{10}\sigma u_1u_2z_1z_2 + d_{01}\gamma v_1v_2z_1z_2 + d_{20}\sigma^2u_1^2u_2^2 + d_{11}\sigma\gamma u_1u_2v_1v_2 + d_{02}\gamma^2v_1^2v_2^2 = 0. \tag{60}$$

The obtained linear equations can be solved using the following procedure. Firstly, solve for b_{11}, b_{20} from (51), (52) and for b_{00}, b_{01} from (55), (56). Then, b_{10} can be obtained from (59). Parameters b_{00}, b_{10}, b_{01}, b_{20}, b_{11} depend on b_{02} which can be chosen arbitrarily. Analogous computations can be repeated for d_{00}, d_{10}, d_{01}, d_{11}, d_{02} and equations (53), (54), (57), (58), (60).

The described derivations demonstrate that system (34), (35) can admit bright/dark solitary solutions.

3.3. Solitary solutions with constraints – case $N = 3$

Let $N = 3$. Then:

$$Z(\widehat{z}) = (\widehat{z} - z_1)(\widehat{z} - z_2)(\widehat{z} - z_3); \quad U(\widehat{z}) = (\widehat{z} - u_1)(\widehat{z} - u_2)(\widehat{z} - u_3); \quad V(\widehat{z}) = (\widehat{z} - v_1)(\widehat{z} - v_2)(\widehat{z} - v_3). \tag{61}$$

For $N = 3$, equations (38), (39) describe sextic polynomials in $\widehat{z}$, thus 14 linear equations are obtained. Taking values $\widehat{z} = z_j, u_j, 0$; $j = 1, 2, 3$ in (38) yields:

$$\eta\sigma z_1(z_1 - z_2)(z_3 - z_1)U(z_1) = b_{20}\sigma^2U^2(z_1) + b_{11}\sigma\gamma U(z_1)V(z_1) + b_{02}\gamma^2V^2(z_1); \tag{62}$$

$$\eta\sigma z_2(z_1 - z_2)(z_2 - z_3)U(z_2) = b_{20}\sigma^2U^2(z_2) + b_{11}\sigma\gamma U(z_2)V(z_2) + b_{02}\gamma^2V^2(z_2); \tag{63}$$

$$\eta\sigma z_3(z_1 - z_3)(z_3 - z_2)U(z_3) = b_{20}\sigma^2U^2(z_3) + b_{11}\sigma\gamma U(z_3)V(z_3) + b_{02}\gamma^2V^2(z_3); \tag{64}$$

$$\eta\sigma u_1(u_1 - u_2)(u_1 - u_3)Z(u_1) = b_{00}Z^2(u_1) + b_{01}\gamma V(u_1)Z(u_1) + b_{02}\gamma^2V^2(u_1); \tag{65}$$

$$\eta\sigma u_2(u_1-u_2)(u_3-u_2)Z(u_2)=b_{00}Z^2(u_2)+b_{01}\gamma V(u_2)Z(u_2)+b_{02}\gamma^2V^2(u_2); \tag{66}$$

$$\eta\sigma u_3(u_1-u_3)(u_2-u_3)Z(u_3)=b_{00}Z^2(u_3)+b_{01}\gamma V(u_3)Z(u_3)+b_{02}\gamma^2V^2(u_3); \tag{67}$$

$$b_{00}z_1^2z_2^2z_3^2+b_{10}\sigma u_1u_2u_3z_1z_2z_3+b_{01}\gamma v_1v_2v_3z_1z_2z_3+b_{20}\sigma^2u_1^2u_2^2u_3^2+b_{11}\sigma\gamma u_1u_2u_3v_1v_2v_3+b_{02}\gamma^2v_1^2v_2^2v_3^2=0. \tag{68}$$

Analogous computations with respect to (39) result in:

$$\eta\gamma z_1(z_1-z_2)(z_3-z_1)V(z_1)=d_{20}\sigma^2U^2(z_1)+d_{11}\sigma\gamma U(z_1)V(z_1)+d_{02}\gamma^2V^2(z_1); \tag{69}$$

$$\eta\gamma z_2(z_1-z_2)(z_2-z_3)V(z_2)=d_{20}\sigma^2U^2(z_2)+d_{11}\sigma\gamma U(z_2)V(z_2)+d_{02}\gamma^2V^2(z_2); \tag{70}$$

$$\eta\gamma z_3(z_1-z_3)(z_3-z_2)V(z_3)=d_{20}\sigma^2U^2(z_3)+d_{11}\sigma\gamma U(z_3)V(z_3)+d_{02}\gamma^2V^2(z_3); \tag{71}$$

$$\eta\gamma v_1(v_1-v_2)(v_1-v_3)Z(v_1)=d_{00}Z^2(v_1)+d_{10}\sigma U(v_1)Z(v_1)+d_{20}\sigma^2U^2(v_1); \tag{72}$$

$$\eta\gamma v_2(v_1-v_2)(v_3-v_2)Z(v_2)=d_{00}Z^2(v_2)+d_{10}\sigma U(v_2)Z(v_2)+d_{20}\sigma^2U^2(v_2); \tag{73}$$

$$\eta\gamma v_3(v_1-v_3)(v_2-v_3)Z(v_3)=d_{00}Z^2(v_3)+d_{10}\sigma U(v_3)Z(v_3)+d_{20}\sigma^2U^2(v_3); \tag{74}$$

$$d_{00}z_1^2z_2^2z_3^2+d_{10}\sigma u_1u_2u_3z_1z_2z_3+d_{01}\gamma v_1v_2v_3z_1z_2z_3+d_{20}\sigma^2u_1^2u_2^2u_3^2+d_{11}\sigma\gamma u_1u_2u_3v_1v_2v_3+d_{02}\gamma^2v_1^2v_2^2v_3^2=0. \tag{75}$$

Note that (62)–(68) has 6 unknowns and 7 equations. Furthermore, the latter equations are not linearly dependent for arbitrary values of z_j, u_j, v_j; $j=1,2,3$. Thus the system is inconsistent if solitary solution parameters are chosen arbitrarily.

Without loss of generality, let $z_3=z_2$. Then (63) and (64) are congruent and read:

$$b_{20}\sigma^2U^2(z_2)+b_{11}\sigma\gamma U(z_2)V(z_2)+b_{02}\gamma^2V^2(z_2)=0. \tag{76}$$

The same constraint also transforms (70) and (71) into:

$$d_{20}\sigma^2U^2(z_2)+d_{11}\sigma\gamma U(z_2)V(z_2)+d_{02}\gamma^2V^2(z_2)=0. \tag{77}$$

Both (62)–(68) and (69)–(75) have nontrivial solutions when $z_3=z_2$. Thus, there exist systems of the form (34), (35) that admit order $N=3$ solitary solutions, but only if the constraint $z_3=z_2$ on solution parameters holds true. Similar derivations can be used to demonstrate that the number and the complexity of constraints grows as N increases.

4. Concluding remarks

Necessary existence conditions for solitary solutions to a class of nonlinear PDEs with multiplicative polynomial coupling have been derived in this paper. While uncoupled PDEs with equivalent nonlinearity can only admit solitary solutions of order $N \le 3$ [21], systems of coupled PDEs have completely different properties. First order coupled systems can admit solitary solutions of order up to $N=13$. Second order systems satisfy necessary existence conditions for solitary solutions of order up to $N=7$. As the order of the system increases, more complex solitary solutions appear (see Figs. 1 and 2).

Some higher order solitary solutions can only satisfy the coupled system with additional constraints on the solution parameters. It is demonstrated that the inverse balancing technique can be used to derive these conditions. It is clear that presented techniques provide a solid computational framework for the application of direct methods for the construction of solitary solutions. A priori consideration of necessary existence conditions greatly reduces the computational complexity of direct methods for the construction of solutions.

Extension of the inverse balancing technique to other systems of PDEs and different forms of analytical solutions remains a definite objective of future research.

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Appendix A. Table of necessary existence conditions

Table 1
Table of necessary existence condition (32) for solitary solutions (7), (8) to (3), (4). $\exists$ denotes existence with all solitary solution parameter values, $\exists^*$ denotes existence with additional constraints on solitary solution parameters, $\nexists$ denotes the nonexistence of solitary solutions.

	$n = m$									
N	1	2	3	4	5	6	7	8	9	10
1	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$
2	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$
3	$\exists^*$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$
4	$\exists^*$	$\exists^*$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$
5	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$
6	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists$	$\exists$	$\exists$	$\exists$
7	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists$
8	$\exists^*$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$
9	$\exists^*$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\exists^*$
10	$\exists^*$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
11	$\exists^*$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
12	$\exists^*$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
13	$\exists^*$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
14	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

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