Weak and strong orders of linear recurring sequences

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Abstract The concept of the strong order of linear recurring sequence (LRS) is introduced in this paper. Necessary and sufficient conditions for the existence of the strong LRS order are derived. The strong LRS order is exploited for the formalization of the problem of the extension of a sequence from the available fragment (fragments) of that sequence. The definition of the strong LRS order opens new possibilities for formal sequence analysis whenever the weak LRS order of that sequence exists. Computational experiments with discrete iterative maps are used to illustrate the applicability of the strong LRS order in nonlinear system analysis.

Keywords Linear recurring sequence · Strong and weak orders · Sequence extrapolation

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1 Introduction

The theory and applications of linear recurrence sequences have been addressed in a large variety of publications during the recent decades. Mikhalev et al. discuss linear recurring sequences over modules (Mikhalev and Nechaev 1996); Kurakin et al. (1995) consider k-dimensional linear recurring sequences over rings and modules; Everest et al. (2003) analyze

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the number-theoretic properties of linear recurrence sequences and their generalizations. Linear recurring sequence subgroups in finite fields are considered in Brison and Nogueira (2003); exponential sums for nonlinear recurring sequences are discussed in Niederreiter and Winterhof (2008); the ABC conjecture in binary recurring sequences is investigated in Ribenboim and Walsh (1999); a completely automated approach to identifying recurring local sequence motifs in proteins is presented in Han and Baker (1995); canonical forms for recurring sequences over Galois field are derived in Singh and Al-Zaid (1991).

Recent developments in error correcting codes and cryptography in general has attracted a new wave of attention to linear recurring sequences (Fillmore and Marx 1968). Sequences defined by periodic recursive relations are investigated in Tam (1997). Linear recurring arrays are used to define and characterize n-dimensional cyclic codes and their dual codes over quasi-Frobenius rings in Lu et al. (2004). The relationship of linear recurrences with polynomial coefficients to sequences generated by recurrences with constant coefficients is discussed in Bronstein and Solé (2004). It is shown that general conditional recurrences satisfy a single recurrence equation in Panario et al. (2014). Linear recurring sequences and subfield subcodes of cyclic codes are analyzed in Gao and Fu (2013).

In this paper, we ask questions: Is it possible to extrapolate any sequence of real numbers? Are these extrapolations unique? Is it possible to extrapolate not only forwards but also backwards? Do these extrapolations depend on the starting element? Providing answers to all these questions is the main objective of this paper.

The primary goal of this paper is to present the concept of the strong and the weak orders of linear recurring sequences (LRS). And though the weak LRS order is introduced in Navickas and Bikulčiene (2006) and is successfully exploited for the construction of solitary solutions to ordinary differential equations in Navickas et al. (2010) and for the exploration of the onset of chaos in Ragulskis and Navickas (2011), we extend and generalize the definition of the LRS order. This definition opens new possibilities for algebraic time series analysis; appropriate derivations and illustrative examples are used to demonstrate the potential of the strong and the weak LRS orders in discrete sequence analysis.

2 Preliminaries

2.1 Singly and doubly infinite sequences

Definition 1 A mapping $P : \mathbb{Z}_0 \to \mathbb{C}$, where $\mathbb{Z}_0 := \mathbb{N} \cup \{0\}$ is called a singly infinite sequence, while a mapping $S : \mathbb{Z} \to \mathbb{C}$ is called a doubly infinite sequence.

The image of any $j \in \mathbb{Z}$ is called the j-th element of a sequence.

We use the notation $P := \left(p_j;\ j \in \mathbb{Z}_0\right)$ and $S := \left(p_j;\ j \in \mathbb{Z}\right)$ for the enumeration of elements of singly and doubly infinite sequences, respectively.

Two sequences $S_1 := \left(p_j;\ j \in \mathbb{Z}\right)$ and $S_2 := \left(q_j;\ j \in \mathbb{Z}\right)$ are equal if and only if

$$p_j = q_j,\ j \in \mathbb{Z}. \tag{1}$$

The central element p_0 of S is identified to avoid misunderstandings. For exam-ple, the sequence $(\ldots, -2, -1, 0, 1, 2, \ldots)$ with $p_0 = 0$ is not equal to the sequence $(\ldots, -2, -1, 0, 1, 2, \ldots)$ where $p_0 = 1$

Standard sequence operations are defined below:

(i) Scalar multiplication:

$$c \cdot S := c\left(p_j;\ j \in \mathbb{Z}\right) := \left(c \cdot p_j;\ j \in \mathbb{Z}\right),\ c \in \mathbb{C}.$$

(ii) Element-wise summation:

$$S_1 + S_2 := \left(p_j;\ j \in \mathbb{Z}\right) + \left(q_j;\ j \in \mathbb{Z}\right) := \left(p_j + q_j;\ j \in \mathbb{Z}\right).$$

(iii) Element-wise multiplication:

$$S_1 \cdot S_2 := \left(p_j;\ j \in \mathbb{Z}\right) \cdot \left(q_j;\ j \in \mathbb{Z}\right) := \left(p_j \cdot q_j;\ j \in \mathbb{Z}\right).$$

(iv) Sub-sequencing:

$$S_n^{(+)} := \left(p_n, p_{n+1}, \ldots\right) := \left(p_{n+j};\ j \in \mathbb{Z}_0\right);$$
$$S_n^{(-)} := \left(\ldots, p_{n-2}, p_{n-1}, p_n\right) := \left(p_{n-j};\ j \in \mathbb{Z}_0\right),$$

where $n \in \mathbb{Z}$ is a fixed index. Sequences $S_n^{(+)}$ and $S_n^{(-)}$ are called subsequences of S.

(v) Element-wise limit: Let $p_j = p_j\left(\lambda_1, \ldots, \lambda_m\right)$, where $\lambda_1, \ldots, \lambda_m$ are function parameters. Then, $S = S\left(\lambda_1, \ldots, \lambda_m\right)$ and

$$\lim_{\lambda_1,\ldots\lambda_m \to \lambda_0} S\left(\lambda_1, \ldots, \lambda_m\right) := \left(\lim_{\lambda_1,\ldots,\lambda_m \to \lambda_0} p_j\left(\lambda_1, \ldots, \lambda_m\right);\quad j \in \mathbb{Z}\right) = S\left(\lambda_0\right).$$

For example, let $p_j\left(\lambda_1, \lambda_2\right) = \frac{\lambda_1^j}{\lambda_1 - \lambda_2} + \frac{\lambda_2^j}{\lambda_2 - \lambda_1},\ j \in \mathbb{Z}$ and $\lambda_1, \lambda_2 \neq 0,\ \lambda_1 \neq \lambda_2$. Then,

$$\begin{aligned}\lim_{\lambda_1,\lambda_2 \to \lambda_0} S(\lambda_1, \lambda_2) &= \left(\lim_{\lambda_1,\lambda_2 \to \lambda_0}\left(\frac{\lambda_1^j}{\lambda_1 - \lambda_2} + \frac{\lambda_2^j}{\lambda_2 - \lambda_1}\right);\ j \in Z\right)\\ &= \left(j\lambda_0^{j-1},\ j \in \mathbb{Z}\right) = S\left(\lambda_0\right),\end{aligned}$$

where $\lambda_0 \neq 0$.

3 The concept and properties of the weak order of LRS

The concept of the linear recurring sequence (LRS) and its properties are introduced in this section.

Suppose a singly infinite sequence $P := \left(p_j;\ j \in \mathbb{Z}_0\right)$ is given. The Hankel mapping of P reads:

$$\mathrm{H}_0\, P := \left(d_j^{(0)};\ j \in \mathbb{Z}_0\right), \tag{2}$$

where

$$d_0^{(0)} := 1, \quad d_n^{(0)} := \det\left(p_{k+l-2}\right)_{1 \le k,l \le n}, \quad n \in \mathbb{Z}_0. \tag{3}$$

Definition 2 The singly infinite sequence $\left(p_j,\ j = 0, 1, 2, \ldots\right)$ is a weak order $m \in \mathbb{Z}_0$ LRS denoted as:

$$\operatorname{order}\left(p_j;\ j \in \mathbb{Z}_0\right) = m, \tag{4}$$

if the Hankel mapping has the following form:

$$\mathrm{H}_0\, P = \left(d_n^{(0)};\, n \in \mathbb{Z}_0\right) = \left(d_0^{(0)}, \ldots, d_m^{(0)}, 0, \ldots\right), \tag{5}$$

where $d_m^{(0)} \neq 0;\; d_{m+k}^{(0)} = 0,\; k = 1, 2, \ldots$

If (5) does not hold for any $m < +\infty$, the sequence is not linearly recurring and the following notation is used:

$$\text{order}\left(p_j;\, j \in \mathbb{Z}_0\right) = +\infty. \tag{6}$$

For example,

1. order $(0, 0, 0, \ldots) = 0$. Note that this is the only sequence with weak order equal to 0;
2. order $(p_0, p_1, \ldots, p_m, 0, 0, \ldots) = m + 1$, where $p_m \neq 0;\; p_{m+k} = 0,\; k = 1, 2, \ldots$ and $m \in \mathbb{N} \ldots$.

Suppose $\left(p_j,\, j = 0, 1, \ldots\right)$ is a LRS and its weak order is $m > 0$. Then it is possible to construct the characteristic equation:

$$\begin{vmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{vmatrix} = 0, \tag{7}$$

which results in a polynomial equation:

$$A_m \rho^m + A_{m-1} \rho^{m-1} + \cdots + A_1 \rho + A_0 = 0, \tag{8}$$

where $A_m = d_m^{(0)} \neq 0$.

Definition 3 The roots $\rho_1, \ldots, \rho_l$ of (8) are called the characteristic roots of $\left(p_j;\, j \in \mathbb{Z}_0\right)$. Multiplicities of ρ_k are denoted by n_k, $k = 1, \ldots, l$. If the characteristic equation has a zero root, the notations ρ_0 and n_0 will be used.

Note that the characteristic equation does not exist for the sequence $(0, 0, 0, \ldots)$.

It has been proven in Navickas et al. (2010) that the sequence $\left(p_j;\, j \in \mathbb{Z}_0\right)$ is a weak order $m \in \mathbb{Z}_0$ LRS:

$$\text{order}\,(p_j,\, j = 0, 1, 2, \ldots) = \sum_{k=0}^{l} n_k = m; \tag{9}$$

if and only if

$$p_j = \sum_{r=0}^{n_0-1} \mu_{0r} \binom{j}{r} 0^{j-r} + \sum_{k=1}^{l} \sum_{r=0}^{n_k-1} \mu_{kr} \binom{j}{r} \rho_k^{j-r}, \tag{10}$$

where $\mu_{0n_0-1}, \mu_{kn_k-1} \neq 0;\;\; k = 1, 2, \ldots, l;\; \binom{j}{k}$ is the binomial coefficient; $0^0 := 1$; $0^k = 0,\; k = 1, 2, \ldots$ It is assumed that $\binom{j}{k}\rho^{j-k} = 0$ if at least one of the multipliers $\binom{j}{k}$ or ρ^k is equal to zero.

If the characteristic equation has no zero roots, (10) simplifies to

$$p_j = \sum_{k=1}^{l} \sum_{r=0}^{n_k-1} \mu_{kr} \binom{j}{r} \rho_k^{j-r}, \tag{11}$$

where $\mu_{kn_k-1} \neq 0$; $k = 1, 2, \ldots, l$.

Remark 1 Note that for any $\rho_k \in \mathbb{C}$ and $\mu_{k,n_k-1} \neq 0$,

$$\text{order}\left(\sum_{r=0}^{n_k-1} \mu_{kr} \binom{j}{r} \rho_k^{j-r};\ j \in \mathbb{Z}_0\right) = n_k. \tag{12}$$

Remark 2 For a singly infinite LRS, the following estimate of the weak order holds for all $n \in \mathbb{Z}_0$:

$$\text{order}\left(p_j;\ j \in \mathbb{Z}_0\right) \geq \text{order}\left(p_{j+n};\ j \in \mathbb{Z}_0\right), \tag{13}$$

because it follows from (10) that characteristic roots equal to zero have a reduced multiplicity in sequence $\left(p_{j+n};\ j \in \mathbb{Z}_0\right)$ for $n > 0$.

Example 1 Suppose a singly infinite sequence $\left(p_j,\ j = 0, 1, \ldots\right)$ is given

$$p_0 = 1, \quad p_1 = 2, p_j = j^2, \quad j = 2, 3, \ldots$$

Then, its Hankel mapping $(d_n, n \in \mathbb{N})$ reads

$$(1, 1, 0, -1,\ 89, 8,\ 0, 0, 0, \ldots).$$

Therefore, order $\left(p_j,\ j \in \mathbb{Z}_0\right) = 5$. The characteristic equation reads

$$\begin{vmatrix} 1 & 2 & 4 & 9 & 16 & 25 \\ 2 & 4 & 9 & 16 & 25 & 36 \\ 4 & 9 & 16 & 25 & 36 & 49 \\ 9 & 16 & 25 & 36 & 49 & 64 \\ 16 & 25 & 36 & 49 & 64 & 81 \\ 1 & \rho & \rho^2 & \rho^3 & \rho^4 & \rho^5 \end{vmatrix} = 0.$$

The arithmetic simplification of the determinant yields

$$8\rho^2 (\rho - 1)^3 = 0.$$

The characteristic roots are $\rho_1 = \rho_2 = 0,\ \rho_3 = \rho_4 = \rho_5 = 1$. Equation (10) yields

$$\begin{aligned} p_j = \mu_{00} \binom{j}{0} 0^j + \mu_{01} \binom{j}{1} 0^{j-1} + \mu_{10} \binom{j}{0} 1^j \\ + \mu_{11} \binom{j}{1} 1^{j-1} + \mu_{12} \binom{j}{2} 1^{j-2}, \end{aligned} \tag{14}$$

where $j = 0, 1 \ldots$

The selection of $j = 0, 1, 2, 3, 4$ in (14) produces a system of linear equations for the determination of coefficients $\mu_{00}, \mu_{01}, \mu_{10}, \mu_{11}, \mu_{12}$. Solving this system produces $\mu_{00} = 1,\ \ \mu_{01} = 1,\ \ \mu_{10} = 0,\ \ \mu_{11} = 1,\ \ \mu_{12} = 2$. Thus, the algebraic expression of the sequence reads

$$p_j = \binom{j}{0} 0^j + \binom{j}{1} 0^{j-1} + \binom{j}{1} 1^{j-1} + 2\binom{j}{2} 1^{j-2},\ \ j = 0, 1, 2, \ldots$$

4 Strong LRS order

4.1 The definition of strong LRS order

Suppose a doubly infinite sequence $S = \left(p_j;\ j \in \mathbb{Z}\right)$ is given. The Hankel matrix of S with center j reads

$$H_n^{(j)} := \left[p_{j+k+l-2}\right]_{1\le k,l\le n}, \tag{15}$$

where $j \in \mathbb{Z}$; $n \in \mathbb{N}$. The Hankel mapping of S with center j reads

$$\mathrm{H}_j S := \left(d_n^{(j)};\ n \in \mathbb{N}\right), \tag{16}$$

where

$$d_0^{(j)} := 1, \quad d_n^{(j)} := \det\left(p_{j+k+l-2}\right)_{1\le k,l\le n},\ n = 1, 2, \ldots \tag{17}$$

Definition 4 The sequence $\left(p_j;\ j \in \mathbb{Z}\right)$ is a strong order $m \in \mathbb{Z}_0$ LRS, denoted as

$$\text{Order}\ \left(p_j;\ j \in \mathbb{Z}\right) = m, \tag{18}$$

if its Hankel mapping has the following form:

$$\mathrm{H}_j\left(p_j;\ j \in \mathbb{Z}\right) = \left(d_n^{(j)};\ n \in \mathbb{Z}_0\right) = \left(1, d_1^{(j)}, \ldots, d_m^{(j)}, 0, \ldots\right), \tag{19}$$

where $d_m^{(j)} \neq 0$; $d_{m+k}^{(j)} = 0,\ k = 1, 2, \ldots$ for all $j \in \mathbb{Z}$ and fixed $m \in \mathbb{N}$. The determinant $d_m^{(j)}$ is the highest order nonzero Hankel determinant of the sequence $(p_j;\ j \in \mathbb{Z})$ where $j \in \mathbb{Z}$ is fixed.

If (19) does not hold for all $j \in \mathbb{Z}$ and some fixed $m \in \mathbb{N}$, the following notation is used

$$\text{Order}\left(p_j;\ j \in \mathbb{Z}\right) = +\infty. \tag{20}$$

Corollary 1 *The following equality holds true:*

$$\text{Order}\ (\ldots, 0, 0, 0, \ldots) = 0, \tag{21}$$

because the Hankel mapping for any $j \in \mathbb{Z}_0$ *reads:* $(1, 0, 0, \ldots)$.

Example 2 Suppose that the elements of $\left(p_j;\ j \in \mathbb{Z}\right)$ are defined as $p_j := \cos\frac{\pi j}{2}$. It can be observed that $p_0 = 1, p_1 = 0, p_2 = -1, p_3 = 0$ and that the sequence is periodic: $p_{j_0+4k} = p_{j_0}$; $j_0 = 0, 1, 2, 3, k \in \mathbb{Z}$.

Note that the Hankel mappings depend on j_0

$$\mathrm{H}_{j_0+4k}\left(p_j;\ j \in \mathbb{Z}\right) = \left(d_n^{(j_0+4k)};\ n \in \mathbb{Z}_0\right) = \left(1, p_{j_0}, -1, 0, \ldots\right); \quad j_0 = 0, 1, 2, 3, k \in \mathbb{Z}, \tag{22}$$

however, the structure of the Hankel mapping satisfies Definition 4, since for all $j = j_0+4k \in \mathbb{Z}$: $d_2^{(j)} = -1 \neq 0$ and $d_3^{(j)} = 0$, thus

$$\text{Order}\ \left(p_j;\ \mathbb{Z}\right) = 2. \tag{23}$$

Example 3 Let $\left(p_j;\ j \in \mathbb{Z}\right) = (\ldots, 1, 1, 2, 2, \ldots)$, where $p_{-j} = 1$, $p_0 = 2$ and $p_j = 2$, for $j = 1, 2, \ldots$. Then, $d_1^{(j)} \neq 0$ for all $j = 0, \pm 1, \ldots$. But $d_2^{(j)} = 0$ only when $j \neq -1, -2$, because $d_2^{(-1)} = -2$, though $d_2^{(-2)} = 1$. Thus,

$$\text{Order}\ (\ldots, 1, 1, 2, 2, \ldots) = +\infty, \tag{24}$$

because the condition $d_m^{(j)} \neq 0, d_{m+k}^{(j)} = 0, k = 1, 2, \ldots$ hold not for all $j \in \mathbb{Z}_0$, however,

$$\text{order}\left(\underbrace{1, 1, \ldots, 1}_{m \text{ times}}, 2, \ldots\right) = m + 1. \tag{25}$$

Suppose a sequence $\left(p_j;\, j \in \mathbb{Z}\right)$ with Order $\left(p_j;\, j \in \mathbb{Z}\right) = m$ is given. The characteristic polynomial reads

$$\widehat{d}_m^{(j)}(\rho) = \begin{vmatrix} p_j & p_{j+1} & \cdots & p_{j+m-1} & p_{j+m} \\ p_{j+1} & p_{j+2} & \cdots & p_{j+m} & p_{j+m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{j+m-1} & p_{j+m} & \cdots & p_{j+2m-2} & p_{j+2m-1} \\ 1 & \rho & \ldots & \rho^{m-1} & \rho^m \end{vmatrix}. \tag{26}$$

Equation $\hat{d}_m^{(j)}(\rho) = 0$ yields

$$\begin{aligned} &a_m^{(j)}(m)\rho^m - a_m^{(j)}(m-1)\rho^{m-1} + (-1)^n a_m^{(j)}(m-n)\rho^{m-n} + \cdots \\ &+ (-1)^m a_m^{(j)}(0) = 0, \end{aligned} \tag{27}$$

where

$$a_m^{(j)}(n) := \begin{vmatrix} p_j & p_{j+1} & \cdots & p_{j+n-1} & p_{j+n+1} & \cdots & p_{j+m} \\ p_{j+1} & p_{j+2} & \cdots & p_{j+n} & p_{j+n+2} & \cdots & p_{j+m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p_{j+m-1} & p_{j+m} & \cdots & p_{j+n+m-2} & p_{j+n+m} & \cdots & p_{j+2m-1} \end{vmatrix}, \tag{28}$$

$n = 0, 1, \ldots, m$. Furthermore, $a_m^{(j)}(m) = d_m^{(j)},\ \ a_m^{(j)}(0) = d_m^{(j+1)}$, and $d_m^{(j)}, d_m^{(j+1)} \neq 0$.

Equation (27) (for a fixed $j \in \mathbb{Z}$) yields the characteristic roots of $\left(p_j;\, j \in \mathbb{Z}\right)$: $\rho_1, \ldots, \rho_l$ with multiplicities $n_1, \ldots, n_l$.

Note that it is impossible to construct the characteristic equation if Order $\left(p_j;\, j \in \mathbb{Z}\right) = 0$ because such LRS has no characteristic roots.

4.2 Main properties of strong LRS order

Let Order $\left(p_j;\, j \in \mathbb{Z}\right) = m$. Then $d_m^{(j)} \neq 0$, but $d_{m+1}^{(j)} = 0$ for all $j \in \mathbb{Z}$.

Consider a situation when elements $p_{j_0}, p_{j_0+1}, \ldots, p_{j_0+2m-1}$ of the sequence $\left(p_j;\, j \in \mathbb{Z}\right)$ are given only for some fixed $j_0 \in \mathbb{Z}$ (all other elements of that sequence are unknown). Then the equality $d_{m+1}^{(j)} = 0$ becomes a linear equation for the determination of p_{j_0+2m}. The element p_{j_0+2m+1} can be determined from the equality $d_{m+1}^{(j+1)} = 0$. The continuation of the process enables to extrapolate all elements $p_{j_0+2m+n-1}$, $n \in \mathbb{N}$, which results in a time-forward extrapolation of the sequence $\left(p_j;\, j \in \mathbb{Z}\right)$. Analogously, linear equations $d_{m+1}^{(j_0-n)} = 0$, $n \in \mathbb{N}$ yield elements p_{j_0-n}, $n \in \mathbb{N}$, which results in a time-backward extrapolation of the given sequence.

Example 4 Let Order $\left(p_j;\, j \in \mathbb{Z}\right) = 3$ and $p_0 = 0$, $p_1 = 1$, $p_2 = 4$, $p_3 = 9$, $p_4 = 16$, $p_5 = 25$.

Then, equalities

$$\begin{vmatrix} p_{-1} & 0 & 1 & 4 \\ 0 & 1 & 4 & 9 \\ 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & 1 & 4 & 9 \\ 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & p_6 \end{vmatrix} = 0,$$

yield $p_{-1} = 1,\ p_6 = 36$. The continuation of the process produces $p_j = j^2$, $j \in \mathbb{Z}$.

The described algorithm is referred to as the first extrapolation algorithm.

Lemma 1 *Let* Order $(p_j;\ j \in \mathbb{Z}) = m$. *Then constants*

$$A_k := (-1)^{m-k-1} \frac{a_m^{(j)}(k)}{a_m^{(j)}(m)}, \quad k = 0, \ldots, m-1 \tag{29}$$

do not depend on j. *Moreover,*

$$p_{j+m} = A_0 p_j + \cdots + A_{m-1} p_{j+m-1}, \tag{30}$$

for $j \in \mathbb{Z}$.

Proof Let Order $(p_j;\ j \in \mathbb{Z}) = m$ and select any $j_0 \in \mathbb{Z}$. Definition 4 yields that $d_{m+1}^{(j_0)} = 0$ or, equivalently

$$\begin{vmatrix} p_{j_0} & p_{j_0+1} & \cdots & p_{j_0+m} \\ p_{j_0+1} & p_{j_0+2} & \cdots & p_{j_0+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{j_0+m} & p_{j_0+m+1} & \cdots & p_{j_0+2m} \end{vmatrix} = 0. \tag{31}$$

The minor corresponding to the first row and column of (31) is equal to $d_m^{(j_0)} \neq 0$ by Definition 4. Thus, the $(m+1)$st column of the matrix corresponding to determinant (31) can be expressed as a linear combinations of the remaining columns

$$p_{j_0+m} = A_0 p_{j_0} + A_1 p_{j_0+1} + \cdots + A_{m-2} p_{j_0+m-2} + A_{m-1} p_{j_0+m-1}; \tag{32}$$

$$p_{j_0+m+1} = A_0 p_{j_0+1} + A_1 p_{j_0+2} + \cdots + A_{m-2} p_{j_0+m-1} + A_{m-1} p_{j_0+m};$$

$$\vdots \tag{33}$$

$$p_{j_0+2m-1} = A_0 p_{j_0+m-1} + A_1 p_{j_0+m} + \cdots + A_{m-2} p_{j_0+2m-3} + A_{m-1} p_{j_0+2m-2}; \tag{34}$$

$$p_{j_0+2m} = A_0 p_{j_0+m} + A_1 p_{j_0+m+1} + \cdots + A_{m-2} p_{j_0+2m-2} + A_{m-1} p_{j_0+2m-1}. \tag{35}$$

Equations (32)–(35) form a system of $m+1$ linear equations with respect to m variables $A_0, \ldots, A_{m-1}$. However, it is enough to consider the first m Eqs. (32)–(34), since multiplying the kth equation by A_k, where $k = 0, \ldots, m-1$ and subtracting from the last equation results in the identity $0 = 0$. Thus, the following relations hold true:

$$p_{j_0+m+k} = A_0 p_{j_0+k} + \cdots + A_{m-1} p_{j_0+m+k-1}, \quad k = 0, \ldots, m-1. \tag{36}$$

The linear system (36) is consistent, since the system's determinant is $d_m^{(j_0)} \neq 0$, thus Cramer's rule yields

$$A_k = (-1)^{m-k-1} \frac{a_m^{(j_0)}(k)}{a_m^{(j_0)}(m)}, \quad k = 0, \ldots, m-1. \tag{37}$$

It remains to prove that coefficients (37) are independent of index j_0. Since the sequence $\left(p_j;\, j \in \mathbb{Z}\right)$ is a strong LRS, the determinant $d_{m+1}^{(j_0+1)} = 0$. Thus, the following linear equations hold true with some constant coefficients $A_0^*, \ldots, A_{m-1}^*$:

$$p_{j_0+m+1} = A_0^* p_{j_0+1} + A_1^* p_{j_0+2} + \cdots + A_{m-2}^* p_{j_0+m-1} + A_{m-1}^* p_{j_0+m}; \tag{38}$$
$$\vdots$$
$$p_{j_0+2m+2} = A_0^* p_{j_0+m+1} + A_1^* p_{j_0+m+2} + \cdots + A_{m-2}^* p_{j_0+2m-1} + A_{m-1}^* p_{j_0+2m}. \tag{39}$$

Subtracting (38) from (33) and continuing this process results into the following set of equations:

$$p_{j_0+k+1}\left(A_0 - A_0^*\right) + p_{j_0+k+2}\left(A_1 - A_1^*\right) + \cdots + p_{j_0+k+m}\left(A_{m-1} - A_{m-1}^*\right) = 0, \tag{40}$$

where $k = 0, \ldots, m-1$. Equation (40) define a homogenous linear system of equations with respect to unknowns $A_k - A_k^*$ that has determinant $d_m^{(j_0+1)} \neq 0$, thus only the trivial solution exists

$$A_k - A_k^* = 0, \quad k = 0, \ldots, m-1. \tag{41}$$

Therefore, the coefficients A_k remain unchanged if j_0 is increased to $j_0 + 1$. The same arguments holds for selecting $j_0 - 1$. Continuing this process proves that coefficients $A_0, \ldots, A_{m-1}$ do not depend on j, which completes the proof. □

Note that $A_0 \neq 0$. Equations (29) and (30) yield

$$a_m^{(j)}(m) p_{j+m} + \cdots + (-1)^k a_m^{(j)}(m-k) p_{j+m-k} + \cdots + (-1)^m a_m^{(j)}(0) p_j = 0. \tag{42}$$

Example 5 Let us consider the sequence $\left(j^2;\, j \in \mathbb{Z}\right)$. The determinant $d_4^{(j_0)} = 0$ ($d_3^{(j_0)} \neq 0$) for all $j_0 \in \mathbb{Z}$ yields constants $A_0 = 1, A_1 = -3, A_2 = 3$. Thus, $(j+3)^2 = j^2 - 3(j+1)^2 + 3(j+2)^2$, for all $j \in \mathbb{Z}$.

Corollary 2 *Suppose that* Order $\left(p_j;\, j \in \mathbb{Z}\right) = m$. *Then there exist constants* $B_k, k = 0, \ldots, m-1$ *independent of* j *such that*

$$p_{j-1} = B_0 p_j + \cdots + B_{m-1} p_{j+m-1}, \tag{43}$$

for all $j \in \mathbb{Z}$.

Proof Equation (30) yields

$$A_0 p_{j-1} = -A_1 p_j - \cdots - A_{m-1} p_{j+m-2} + p_{j+m-1}. \tag{44}$$

Then (44) results in (43) with

$$B_k = -\frac{A_{k+1}}{A_0}, \quad k = 0, \ldots, m-2; \quad B_{m-1} = \frac{1}{A_0} \neq 0. \tag{45}$$

□

Example 6 Let us consider the sequence $\left(j^2;\, j \in \mathbb{Z}\right)$. Then conditions $d_{j_0-1}^{(4)} = 0$ and $d_{j_0-1}^{(3)} \neq 0$ (for all $j_0 \in \mathbb{Z}$) yield $B_0 = 3, B_1 = -3, B_2 = 1$. Thus, $(j-1)^2 = 3j^2 - 3(j+1)^2 + (j+2)^2$, for all $j \in \mathbb{Z}$.

Corollary 3 *Let* Order $\left(p_j;\, j \in \mathbb{Z}\right) = m$. *Then the characteristic equation* (27) *is reducible to*

$$\rho^m - A_{m-1}\rho^{m-1} - \cdots - A_1\rho - A_0 = 0, \tag{46}$$

where coefficients $A_0, \ldots, A_{m-1}$ *are defined in* (29).

Proof The proof follows from (27) and (29). □

5 Algebraic progressions

5.1 Ordinary algebraic progression

Definition 5 Suppose $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_m$ are constants that satisfy conditions

$$\lambda_k, \mu_k \in \mathbb{C}/\{0\}; \quad \lambda_k \neq \lambda_l,\ k \neq l. \tag{47}$$

Then the sequence $\left(q_j;\, j \in \mathbb{Z}\right)$

$$q_j := \sum_{k=1}^{m} \mu_k \lambda_k^j,\ j \in \mathbb{Z}; \tag{48}$$

is an ordinary algebraic progression.

In the following theorem, we denote the Vandermonde determinant as

$$V_m\left(\lambda_1, \ldots, \lambda_m\right) := \det\left(\lambda_{i+1}^j\right)_{0 \le i, j \le m-1} = \prod_{r<l}\left(\lambda_l - \lambda_r\right). \tag{49}$$

Theorem 1 *The sequence defined by* (48) *is a strong order LRS with*

$$\text{Order}\left(q_j;\, j \in \mathbb{Z}\right) = m. \tag{50}$$

The sequence's highest order nonzero Hankel determinant $d_m^{(j)}$ *takes the form*

$$d_m^{(j)} = \left(V_m\left(\lambda_1, \ldots, \lambda_m\right)\right)^2 \prod_{l=1}^{m} \mu_l \lambda_l^j; \tag{51}$$

and the characteristic roots read

$$\rho_k = \lambda_k, \quad k = 1, \ldots, m. \tag{52}$$

The detailed proof of the theorem is given in Appendix A.

Suppose that Order $\left(p_j;\, j \in \mathbb{Z}\right) = m < +\infty$. Moreover, let all characteristic roots of this sequence $\rho_1, \ldots, \rho_m$ be different

$$\rho_k \neq \rho_l, \quad k \neq l; \quad k, l = 1, \ldots, m. \tag{53}$$

Then it is possible to construct the sequence $\left(q_j;\, j \in \mathbb{Z}\right)$ using (48) by assuming that $\lambda_k = \rho_k,\ k = 1, \ldots, m$ and selecting such coefficients μ_k that conditions

$$q_j = p_j, \quad j = 0, \ldots, m-1, \tag{54}$$

hold true. Such reconstruction is possible because the system of linear Eq. (54) has only one solution—the determinant of this system of equations (for the determination of unknowns

μ_r, $r = 1, \ldots, m$) is the nonzero Vandermonde determinant, $V_m(\lambda_1, \ldots, \lambda_m) \neq 0$. Note that there cannot exist such $k \in \{1, \ldots, m\}$ that $\mu_k = 0$. Suppose that there exists $\mu_k = 0$ for some $k \in \{1, \ldots, m\}$. Then, (51) yields that $d_m^{(j)} = 0$ for the sequence $(p_j;\ j \in \mathbb{Z})$, which violates the assumption that $(p_j;\ j \in \mathbb{Z})$ is a strong order m LRS.

The sequence $(q_j;\ j \in \mathbb{Z})$ is denoted as the secondary sequence generated by the strong order $m < +\infty$ LRS $(p_j;\ j \in \mathbb{Z})$. The algorithm for the construction of this secondary sequence is called the second extrapolation algorithm.

Lemma 2 *Suppose a sequence of numbers $(p_j;\ j \in \mathbb{Z})$ with* Order $(p_j;\ j \in \mathbb{Z}) = m < +\infty$ *is given and its characteristic roots satisfy* (46). *Then the sequence $(q_j;\ j \in \mathbb{Z})$ constructed using the second extrapolation algorithm coincides with the sequence $(p_j;\ j \in \mathbb{Z})$*

$$p_j = q_j, \quad j \in \mathbb{Z}. \tag{55}$$

Proof Both sequences $(p_j;\ j \in \mathbb{Z})$ and $(q_j;\ j \in \mathbb{Z})$ are strong order LRS of equal order and possess equal characteristic roots (after necessary reordering).

Coefficients $A_0, \ldots, A_{m-1}$ of sequences $(p_j;\ j \in \mathbb{Z})$ and $(q_j;\ j \in \mathbb{Z})$ defined by (29) also coincide. Equation (54) holds because (42) holds true. Therefore, (55) holds. □

Corollary 4 *Elements p_j of the sequence $(p_j;\ j \in \mathbb{Z})$ can be expressed in the form*

$$p_j = \sum_{k=1}^{m} \mu_k \rho_k^j, \quad j \in \mathbb{Z}. \tag{56}$$

if and only if the sequence is a strong order m LRS and all characteristic roots ρ_k, $k = 1, \ldots, m$ are nonzero and distinct.

5.2 Binomial algebraic progression

Definition 6 A sequence $(p_j;\ j \in \mathbb{Z})$ is a binomial algebraic progression if its elements are expressed in the form

$$p_j = a \frac{j(j-1)\cdots(j-m+2)}{(m-1)!} \lambda_0^{j-m+1}, \tag{57}$$

where $m \in \mathbb{Z}_0;\ a, \lambda_0 \in \mathbb{C}/\{0\}$.

In this section, it will be shown that (57) is a strong order LRS.

Suppose that $\lambda_k \in \mathbb{C}/\{0\}\ k = 1, \ldots, m$ satisfy conditions $\lambda_k \neq \lambda_l$, $k \neq l$. Then it is possible to construct determinants $\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m)$

$$\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m) := \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{m-2} & \lambda_1^j \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{m-2} & \lambda_2^j \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \ldots & \lambda_m^{m-2} & \lambda_m^j \end{vmatrix}, \tag{58}$$

where $m = 2, 3, \ldots;\ j \in \mathbb{Z}$.

The following equalities hold

$$\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m) = 0, \quad j = 0, \ldots, m-2; \tag{59}$$

$$\overline{V}_m^{(m-1)}(\lambda_1, \ldots, \lambda_m) = V_m(\lambda_1, \ldots, \lambda_m). \tag{60}$$

Expanding the determinant (58) by the last column yields

$$\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m) = \sum_{k=1}^{m} (-1)^{m+k} \lambda_k^j V_{m-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m), \tag{61}$$

where $j = 0, \pm 1, \ldots$

Using (58), the sequence of numbers $S_m(\lambda_1, \ldots, \lambda_m)$ is defined as

$$S_m(\lambda_1, \ldots, \lambda_m) = \left(\Delta_m^{(j)}(\lambda_1, \ldots, \lambda_m); \, j \in \mathbb{Z}\right), \tag{62}$$

where

$$\Delta_m^{(j)}(\lambda_1, \ldots, \lambda_m) := a \frac{\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m)}{V_m(\lambda_1, \ldots, \lambda_m)}, \tag{63}$$

and $a \neq 0$ is a fixed complex number. Equations (61) and (63) yield

$$\begin{aligned} \Delta_m^{(j)}(\lambda_1, \ldots, \lambda_m) &= \sum_{k=1}^{m} (-1)^{k+m} a \frac{V_{m-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m)}{V_m(\lambda_1, \ldots, \lambda_m)} \lambda_k^j \\ &= \sum_{k=1}^{m} \nu_k \lambda_k^j. \end{aligned} \tag{64}$$

Thus, the sequence (62) is an algebraic progression and its coefficients $\nu_k = \nu_k(\lambda_1, \lambda_2, \ldots, \lambda_m)$ depend on parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$. Moreover, the following equalities hold true

(i)

$$\nu_k = (-1)^{k+m} a \frac{V_{m-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_m)}{V_m(\lambda_1, \ldots, \lambda_m)} = \frac{a}{\prod_{\substack{l=1,\ldots,m \\ l \neq k}} (\lambda_k - \lambda_l)}; \tag{65}$$

(ii)

$$\text{Order } S_m(\lambda_1, \ldots, \lambda_m) = m. \tag{66}$$

(iii) The highest order nonzero Hankel determinant $d_m^{(j)}(S_m)$ of the sequence $S_m(\lambda_1, \ldots, \lambda_m)$ can be expressed in the form

$$d_m^{(j)}(S_m) = (-1)^{\binom{m}{2}} a^m \prod_{k=1}^{m} \lambda_k^j, \tag{67}$$

because

$$\prod_{k=1}^{m} \nu_k = \frac{a^m}{V_m^2(\lambda_1, \lambda_2, \ldots, \lambda_m)}. \tag{68}$$

Theorem 2 *The sequence* $\widehat{S}_m(\lambda_0) = \left(p_j; \, j \in \mathbb{Z}\right)$ *is a strong order LRS*

$$\text{Order } \widehat{S}_m(\lambda_0) = m, \tag{69}$$

if

$$p_j = a \frac{j(j-1)\cdots(j-m+2)}{(m-1)!} \lambda_0^{j-m+1}, \tag{70}$$

where $m = 2, 3, \ldots$; $j \in \mathbb{Z}_0$ *and* $a, \lambda_0 \in \mathbb{C}/\{0\}$. *Moreover, the highest order nonzero Hankel determinant of the sequence* $\widehat{S}_m(\lambda_0)$ *reads*

$$d_m^{(j)}\left(\widehat{S}_m\right) = (-1)^{\binom{m}{2}} a^m \lambda_0^{mj} \neq 0. \tag{71}$$

Proof The L'Hopital rule and formulas for the differentiation of determinants (Appendix B) yield the equality

$$\lim_{\lambda_1,\ldots,\lambda_m\to\lambda_0} \Delta_m^{(j)}(\lambda_1,\ldots,\lambda_m) = a\frac{j(j-1)\ldots(j-m+2)}{(m-1)!}\lambda_0^{j-m+1}. \tag{72}$$

Equation (67) yields

$$\lim_{\lambda_1,\ldots,\lambda_m\to\lambda_0} d_m^{(j)}(S_m) = (-1)^{\binom{m}{2}} a^m \lambda_0^{mj} := d_j^{(m)}\left(\widehat{S}_m\right) \neq 0. \tag{73}$$

On the other hand,

$$d_{m+n}^{(j)}(S_m) = d_j^{(m+n)}\left(\widehat{S}_m\right) = 0;\ n = 1, 2, \ldots \tag{74}$$

Thus,

$$\lim_{\lambda_1,\ldots,\lambda_m\to\lambda_0} S_m(\lambda_1,\ldots,\lambda_m) = \widehat{S}_m(\lambda_0). \tag{75}$$

□

Corollary 5 *The characteristic polynomial of the sequence $\widehat{S}_m(\lambda_0)$ has only one root and its multiplicity is equal to m.*

The proof follows from the definition of the characteristic polynomial (26) and the limit transition from the sequence $S_m(\lambda_1,\ldots,\lambda_m)$ to the sequence $\widehat{S}_m(\lambda_0)$.

The expression of binomial coefficients $\binom{j}{k}$ can be extended for $j = -1, -2, \ldots$ but then the Riordan "negative factorial" equality Riordan (1968) must be used

$$\frac{(-l)!}{(-k)!} = (-1)^{l-k}\frac{(k-1)!}{(l-1)!};\quad l, k = 1, 2, \ldots \tag{76}$$

Let $j = 1, 2, \ldots;\ m = 2, 3, \ldots$ Then

$$\begin{aligned}\binom{-j}{m-1} &= \frac{(-j)!}{(m-1)!(-j-m+1)!} = (-1)^{m-1}\frac{(j+m-2)!}{(m-1)!(j-1)!}\\ &= \frac{(-j)(-j-1)\ldots(-j-m+2)}{(m-1)!}.\end{aligned} \tag{77}$$

Analogously, $\binom{j}{0} = 1$, for all $j \in \mathbb{Z}$.Thus,

$$\widehat{S}_m(\lambda_0) = \left(a\binom{j}{m-1}\lambda_0^{j-m+1};\ j \in \mathbb{Z}\right);\quad m = 2, 3, \ldots \tag{78}$$

5.3 Generalized binomial algebraic progression

Definition 7 Suppose $\lambda_1,\ldots,\lambda_m;\ \mu_1,\ldots,\mu_m$ satisfy (47). A sequence $\left(p_j;\ j \in \mathbb{Z}\right)$ is a generalized binomial algebraic progression if its elements p_j are expressed in the following form:

$$p_j = \sum_{k=1}^{m} \mu_k \binom{j}{k-1}\rho_0^{j-k+1}, \tag{79}$$

where $\rho_0 \in \mathbb{C}/\{0\}$.

Theorem 3 *The sequence defined by* (79) *is a strong order LRS with*

$$\text{Order}\ \left(p_j;\ j \in \mathbb{Z}\right) = m. \tag{80}$$

The sequence's highest order nonzero Hankel determinant takes the form

$$d_m^{(j)} = (-1)^{\binom{m}{2}} \mu_m^m \rho_0^{mj}. \tag{81}$$

Detailed derivations of the theorem's results are given in Appendix C. Note that ρ_0 is the root (with multiplicity m) of the following polynomial equation:

$$\begin{vmatrix} p_j & p_{j+1} & \cdots & p_{j+m} \\ p_{j+1} & p_{j+2} & \cdots & p_{j+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{j+m-1} & p_{j+m} & \cdots & p_{j+2m-1} \\ 1 & \rho & \ldots & \rho^m \end{vmatrix} = 0. \tag{82}$$

The second extrapolation algorithm now yields the following corollary.

Corollary 6 *Elements of the sequence* $\left(p_j;\ j \in \mathbb{Z}\right)$ *can be expressed in the form*

$$p_j = \sum_{k=1}^{m} \mu_k \binom{j}{k-1} \rho_0^{j-k+1}, \tag{83}$$

if and only if

$$\text{Order}\ \left(p_j;\ j \in \mathbb{Z}\right) = m, \tag{84}$$

and characteristic roots are $\rho_1 = \cdots = \rho_m =: \rho_0 \neq 0.$

5.4 Algebraic progressions and their properties

Definition 8 A sequence $\left(p_j;\ j \in \mathbb{Z}\right)$ is an algebraic progression if its elements p_j are expressed in the form

$$p_j = \sum_{k=1}^{m} \sum_{r=0}^{n_k-1} \mu_{kr} \binom{j}{r} \lambda_k^{j-r}, \tag{85}$$

where $\lambda_k,\ \mu_{kn_k-1} \neq 0;\ \lambda_k \neq \lambda_l$ for all $k \neq l$.

Theorem 4 *Elements* p_j *of the sequence* $\left(p_j;\ j \in \mathbb{Z}\right)$ *can be expressed in the form*

$$p_j = \sum_{k=1}^{m} \sum_{r=0}^{n_k-1} \mu_{kr} \binom{j}{r} \lambda_k^{j-r}, \tag{86}$$

where $\lambda_k,\ \mu_{kn_k-1} \neq 0;\ \lambda_k \neq \lambda_l$ *for all* $k \neq l$ *if and only if*

$$\text{Order}\ \left(p_j;\ j \in \mathbb{Z}\right) = \overline{m}, \tag{87}$$

and the multiplicities of its characteristic roots λ_k *are* n_k*;* $k = 1, \ldots, \overline{m}$*, where*

$$n_1 + \cdots + n_{\overline{m}} = \overline{m}. \tag{88}$$

The proof of this theorem is analogous to the proof of Corollaries 5 and 6.

Corollary 7 *The ordinary algebraic progression, the binomial algebraic progression and the generalized binomial algebraic progression are algebraic progressions.*

Note that it is possible to use the "backward sampling" matrix

$$\widetilde{H}_j^{(m)} := \begin{bmatrix} p_j & p_{j-1} & \cdots & p_{j-m+1} \\ p_{j-1} & p_{j-2} & \cdots & p_{j-m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{j-m+1} & p_{j-m} & \cdots & p_{j-2m+2} \end{bmatrix}, \tag{89}$$

instead of the matrix in (15) for the determination of strong LRS order—the property of transpose of the determinant yields the equality

$$\det \widetilde{H}_j^{(m)} = \det H_{j-2m+2}^{(m)}. \tag{90}$$

In that case the characteristic polynomial takes the form

$$\widetilde{d}_j^{(m)} := \begin{vmatrix} \widetilde{\rho}^m & \widetilde{\rho}^{m-1} & \cdots & \widetilde{\rho} & 1 \\ p_{j-1} & p_{j-2} & \cdots & p_{j-m} & p_{j-m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{j-m} & p_{j-m-1} & \cdots & p_{j-2m+1} & p_{j-2m} \end{vmatrix}. \tag{91}$$

This determinant coincides with $\overline{d}_{j-2m}^{(m)}$. Thus, the roots $\overline{\rho}_k$, $k = 1, \ldots, m$ of $\overline{d}_{j-2m}^{(m)} = 0$ satisfy the equality

$$\overline{\rho}_k = \rho_k, \quad k = 1, \ldots, m, \tag{92}$$

where ρ_k are the roots of (27).

Corollary 8 *Suppose elements p_j of the sequence $\left(p_j;\ j \in \mathbb{Z}\right)$ are*

$$p_j = \sum_{k=1}^{m} \sum_{r=0}^{n_k-1} \widehat{\mu}_{kr} \frac{j^r}{r!} \lambda_k^j, \tag{93}$$

where $n_k \in \mathbb{N}$; $\widehat{\mu}_{kn_k-1}$; $\lambda_1, \ldots, \lambda_m \neq 0$; $\lambda_k \neq \lambda_l$ for $k \neq l$ and coefficients $\widehat{\mu}_{kr}$ do not depend on j. Then

$$\text{Order}\ \left(p_j;\ j \in \mathbb{Z}\right) = n_1 + \cdots + n_m. \tag{94}$$

Proof The proof follows from the fact that ordinary algebraic transformations can be used for the rearrangement of (86) into (93). And, on the other hand, (93) can be rearranged into (86). Therefore, both equations describe the same algebraic progression. □

Example 7 Suppose that Order $\left(p_j;\ j \in \mathbb{Z}\right) = 3$ and

$$p_j = \mu_1 \binom{j}{0} \lambda^j + \mu_2 \binom{j}{1} \lambda^{j-1} + \mu_3 \binom{j}{2} \lambda^{j-2}, \tag{95}$$

where $\mu_3, \lambda \neq 0$; $j \in \mathbb{Z}$. Then

$$\begin{aligned} p_j &= \mu_1 \lambda^j + \mu_2 j \lambda^{j-1} + \mu_3 \frac{j(j-1)}{2} \lambda^{j-2} = \mu_1 \lambda^j + \frac{\mu_2}{\lambda} j \lambda^j + \frac{\mu_3}{2\lambda^2} j^2 \lambda^j - \frac{\mu_3}{2\lambda^2} j \lambda^j \\ &= \left(\hat{\mu}_1 + \hat{\mu}_2 j + \hat{\mu}_3 j^2\right) \lambda^j, \end{aligned}$$

where $\hat{\mu}_1 = \mu_1$, $\hat{\mu}_2 = \frac{\mu_2}{\lambda} - \frac{\mu_3}{2\lambda^2}$, $\hat{\mu}_3 = \frac{\mu_3}{2\lambda^2}$.

Let Order $S_1 = m_1$; Order $S_2 = m_2$; $m_1, m_2 \in \mathbb{Z}_0$, where $S_1 := \left(p_j;\ j \in \mathbb{Z}\right)$; $S_2 := \left(q_j;\ j \in \mathbb{Z}\right)$. Then (86) and(94) yield the following properties of algebraic progressions:

(i) Order $c \cdot S_1 =$ Order S_1, where $c \in \mathbb{C}/\{0\}$;
(ii) $0 \le$ Order $(S_1 + S_2) \le m_1 + m_2$;
(iii) $0 \le$ Order $S_1 \cdot S_2 \le m_1 \cdot m_2$;

Let $k \in \mathbb{Z}$. Then

$$\text{Order } \left(p_{j+k};\ j \in \mathbb{Z}\right) = \text{Order } \left(p_j;\ j \in \mathbb{Z}\right). \tag{96}$$

Therefore, sequences cS_1; $S_1 + S_2$; $S_1 \cdot S_2$; $\left(p_{j+k};\ j \in \mathbb{Z}\right)$ are algebraic progressions as well.

Equation (96) illustrates one of the main differences between weak and strong LRS order: if the LRS has a strong order, the Hankel mapping may be formed and the characteristic roots computed from any starting element p_j, $j \in \mathbb{Z}$, which is not the case for weak LRS order (Theorem 2). Also, the order of subsequences is equal to the order of the sequence for strong LRS, which is not the case weak LRS order. Another difference between strong and weak LRS is the fact that the latter may possess a root $\rho_k = 0$ and the former cannot.

6 Computational experiments

This section is devoted for the illustration of the role of the weak and the strong orders in computational analysis of nonlinear dynamical systems.

The procedure for computing the LRS order can be applied to discrete chaotic maps. Even though the sequences generated by these maps are not linear recurrences, the Hankel mapping yields an indicator of the sequence's complexity. It is demonstrated in Ragulskis and Navickas (2011) that linear recurring sequences can be used as a numerical tool for the qualitative investigation of discrete chaotic maps.

Let us consider a discrete iterative map $x_{n+1} = f(x_n)$. A singly infinite sequence $(x_0, x_1, \dots)$ can be formed starting from the initial condition x_0. The Hankel mapping of this sequence reads: $(1, d_1^{(0)}, d_2^{(0)}, \dots)$. Though in general, the sequence $(x_0, x_1, \dots)$ is not a LRS, the pseudo order of this sequence can be computed using the following procedure Landauskas et al. (2016):

1. Choose the upper bound of the pseudo order $M \in \mathbb{N}$ and $\varepsilon > 0$.
2. Use SVD decomposition to compute singular values $\sigma_1, \dots, \sigma_M$; $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_M$ of the Hankel matrix H_M corresponding to the sequence $(x_0, x_1, \dots, x_n, \dots)$.
3. If there exists σ_k such that $\sigma_{k-1} \ge \varepsilon$, $\sigma_k < \varepsilon$ then the pseudo order of the sequence is equal to k.

The concept of the weak and the strong LRS order is illustrated using the Gauss iterated map (also known as the Mouse map) $x_{n+1} = f(x_n)$, given by

$$f(x_n) = \exp\left(-\alpha x_n^2\right) + \beta, \tag{97}$$

where α, β are real parameters. The bifurcation diagram of (97) is shown in Fig. 1; note that two separate period-1 stable attractors coexists in the region $\beta \in (-1, -0.852)$. One of the period-1 stable attractors experiences a period doubling bifurcation at $\beta = -0.852$. The other period-1 stable attractor still coexists in the interval $\beta \in (-0.852, -0.781)$.

The inverse mapping of f is a two-valued function

$$f^{-1}(x_{n+1}) = \pm\sqrt{-\frac{\log(x_{n+1} - \beta)}{\alpha}}. \tag{98}$$

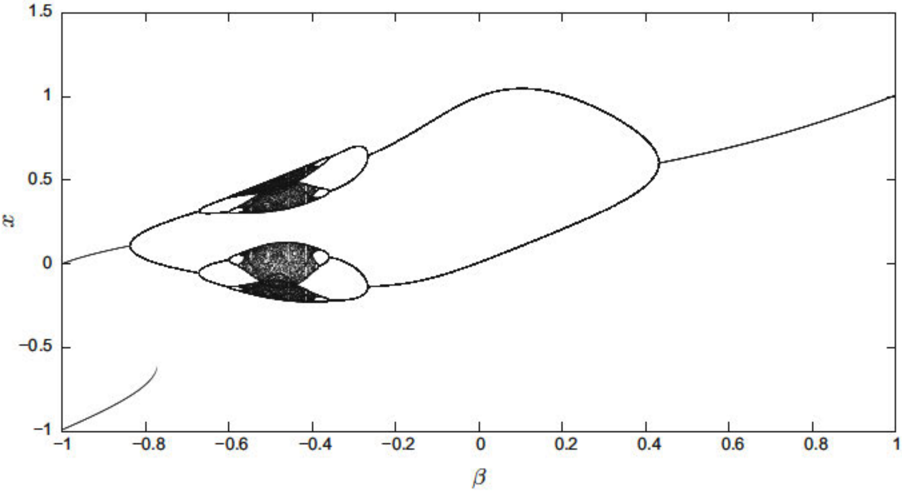

Fig. 1 The bifurcation diagram of the Gauss iterated map with $\alpha = 4.9$

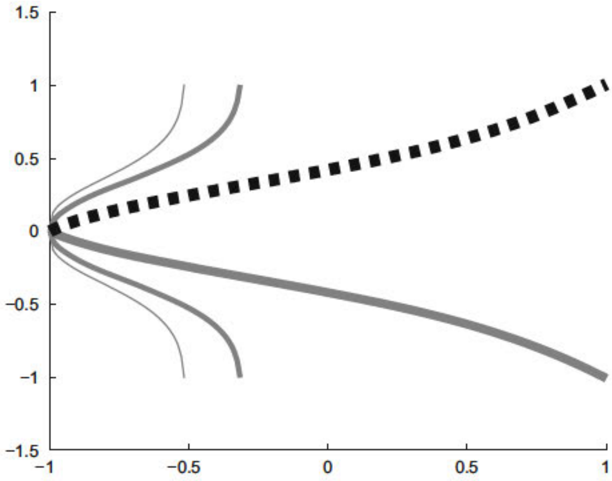

Fig. 2 The NAC manifolds for the period-1 fixed point of the Gauss map at $\alpha = 4.9$. The dashed black line denotes the fixed point, the 5, 3, 1 pixel gray lines denote points that converge non-asymptotically to x_* in 1,2,3 iterations, respectively

Note that the inverse mapping is valid only for $\beta < x_{n+1} \le \beta + 1$ at $\alpha > 0$. Let x_* be a fixed point (stable or unstable) of (97). If $\beta < x_* \le \beta + 1$ a backward iteration defined by (98) yields the values $x_*^{(1)}, x_*^{(2)}$, where $x_*^{(1)} = x_*$ and the remaining point $x_*^{(2)}$ belongs to the non-asymptotic convergence (NAC) manifold of x_* (a set of points that reach the stable or the unstable manifold in a finite number of forward steps). The NAC manifolds for the Gauss iterated map are depicted in Fig. 2. The thick dashed line represents the period-1 fixed point of the Gauss map at $\alpha = 4.9$. Note that period-1 fixed point is a stable attractor at $\beta \in (-1, -0.852)$ and at $\beta \in (0.412, 1)$—but is a repeller at $\beta \in (-0.852, 0.412)$. Nonetheless a sequence started at $x_0 = x_*$ for any $\beta \in (-1, 1)$ results into a period-1 sequence. The thick gray solid line in Fig. 2 corresponds to a set of points which converge non-

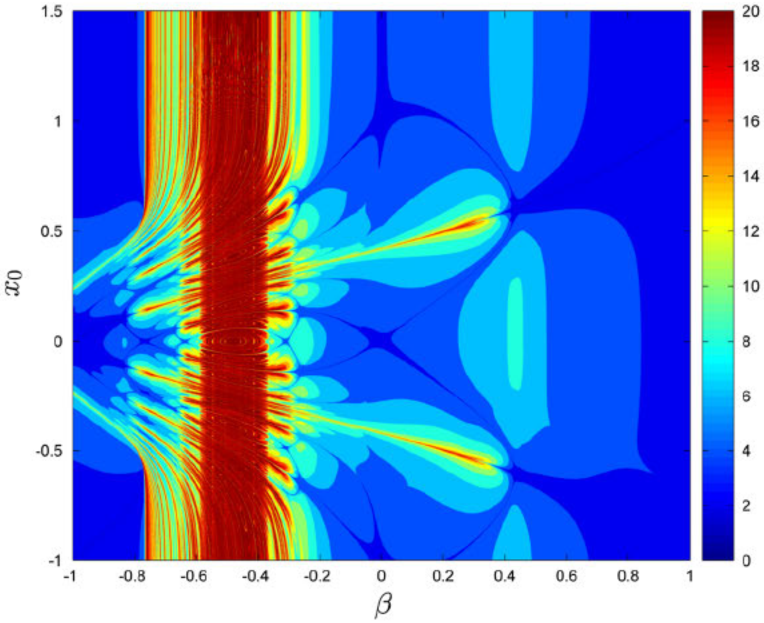

Fig. 3 The map of pseudo orders of the Gauss map without skipping transient processes. $\alpha = 4.9$, $M = 20$

asymptotically to x_* in a single forward step. Similarly, other solid lines in Fig. 2 correspond to sets of points which converge non-asymptotically to x_* in 2 and 3 forward iterations, respectively.

The map of pseudo orders of the Gauss iterated map is shown in Fig. 3. The color of every point in this map corresponds to the LRS pseudo order computed for a sequence $(x_0, x_1, \dots)$ at $\alpha = 4.9$ and β corresponding to the coordinate of that point. The resolution of Fig. 3 is 1000×1000; the procedure for the computation of LRS pseudo order was repeated 10^6 times (total computing time was 60 min on Intel i5 processor). It is interesting to observe the similarity of the NAC manifold and the map of LRS pseudo orders. It is clear that the complexity of a process with short transients is lower compared to long transients and chaotic processes. The map of LRS pseudo orders can be effectively used for a straightforward identification of stable, unstable and NAC manifolds.

Next, we will compute the map of strong pseudo orders for the Gauss map. Previous derivations imply that the computation of a strong pseudo order requires skipping all transient processes in a discrete recurrent map. Thus, the computation procedure is exactly the same as for the map of weak pseudo orders (Fig. 3) except that every point in Fig. 4 denotes an initial condition—but subsequence used for the formation of the Hankel matrix H_M starts after omitting ten thousand forward steps from x_0.

The map of strong pseudo orders can be conveniently used for the identification of the system's sensitivity to initial conditions. Cascades of forward and reverse period doubling bifurcations result into chaotic orbits (Fig 1). The sensitivity to infinitesimal perturbations of initial conditions is one of the qualitative characteristics of deterministic chaos Ott (2002). The map of strong pseudo orders clearly visualizes the chaotic region where a slightest variation of x_0 results into a different pseudo order. The intervals of β corresponding to deterministic orbits are visualized by vertical bands of constant strong pseudo orders. In other words, the global attractor of the system forgets its initial conditions.

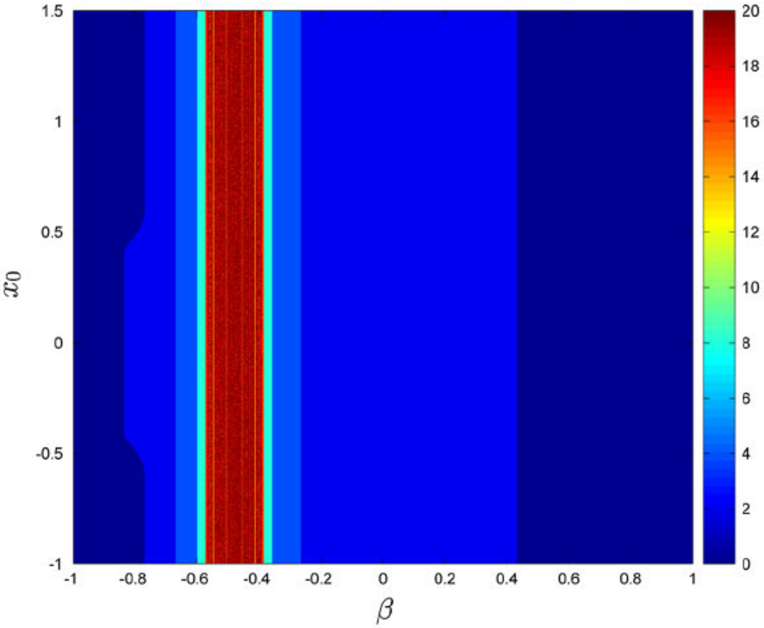

Fig. 4 The map of strong pseudo orders of the Gauss map after skipping transient processes ($j = 10^4$ forward steps). $\alpha = 4.9$, $M = 20$

Interesting phenomena can be observed in the interval $\beta \in (-0.852, -0.781)$ in Fig. 4. As mentioned previously, two different stable attractors coexist in the interval $\beta \in (-0.852, -0.781)$. The map of strong pseudo orders does not only show the existence of these two coexisting attractors—it also determines the basin boundaries of these attractors.

7 Concluding remarks

The concept of the strong LRS order is introduced in this paper. As mentioned previously, the weak LRS order has been previously exploited for the construction of partial solutions of ordinary differential equations, for the exploration of the onset of chaos and for the construction of the manifold of nonasymptotic convergence in discrete nonlinear dynamical systems. This paper provides the derivation of necessary and sufficient conditions for the existence of the strong LRS order and emphasizes its importance in algebraic sequence analysis.

The definition of the strong LRS order opens new possibilities for formal sequence analysis whenever the weak order of that sequence exists. The strong LRS order formalizes the problem of the extension of a sequence from the available fragment (fragments) of that sequence. Moreover, the strong LRS order can be exploited to assess the complexity of nonlinear discrete iterative maps, to identify manifolds and basin boundaries of coexisting attractors. Applicability of the strong order for the construction of closed form solutions to ordinary differential equations and for algebraic analysis of real-world time series remains a definite object of future research.

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Appendix A: Proof of Theorem 1

The definition of the determinant using combinatorial transposition reads [Bernstein (2009), p. 103]

$$\det A := \sum_{(k_1,\ldots,k_m)} (-1)^{\gamma(k_1,\ldots,k_m)} \prod_{r=1}^{m} A_{r,k_r}, \tag{99}$$

where $(k_1, \ldots, k_m)$ is a permutation of the set $\{1, \ldots, m\}$; $\gamma(k_1, \ldots, k_m)$ is the number of transpositions in the permutation $(k_1, \ldots, k_m)$ and A_{r,k_r} is the (r, k_r)-th entry of the m-th order matrix A.

The determinant property [Bernstein (2009), p. 104] $\det\left(A + ve_k^T\right) = \det A + \det A^{(k)}$, where $v \in \mathbb{C}^m$; $e_k \in \mathbb{C}^m$ is a unit vector with all elements except the k-th equal to zero; $A^{(k)}$ is the matrix A with the k-th column replaced by v and (51) yield

$$\begin{aligned} d_m^{(j)} &= \sum_{(k_1,\ldots,k_m)} \left(\prod_{l=1}^{m} \mu_l \lambda_l^j\right) \begin{vmatrix} 1 & \lambda_{k_1} & \lambda_{k_1}^2 & \ldots & \lambda_{k_1}^{m-1} \\ \lambda_{k_2} & \lambda_{k_2}^2 & \lambda_{k_2}^3 & \ldots & \lambda_{k_2}^{m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{k_m}^{m-1} & \lambda_{k_m}^{m} & \lambda_{k_m}^{m+1} & \ldots & \lambda_{k_m}^{2m-2} \end{vmatrix} \\ &= \left(\prod_{l=1}^{m} \mu_l \lambda_l^j\right) \sum_{(k_1,k_2,\ldots,k_m)} (-1)^{\gamma(k_1,\ldots,k_m)} V_m(\lambda_1, \ldots, \lambda_m) \prod_{r=1}^{m} \lambda_{k_r}^{r-1} \\ &= (V_m(\lambda_1, \ldots, \lambda_m))^2 \prod_{l=1}^{m} \mu_l \lambda_l^j, \end{aligned} \tag{100}$$

because (99) yields

$$V_m(\lambda_1, \ldots, \lambda_m) = \sum_{(k_1,k_2,\ldots,k_m)} (-1)^{\gamma(k_1,\ldots,k_m)} \prod_{r=1}^{m} \lambda_{k_r}^{r-1}. \tag{101}$$

Analogous rearrangements let us prove that

$$d_j^{(m+1)} = 0, \quad j \in \mathbb{Z}. \tag{102}$$

Therefore, (50) holds true. On the other hand, it is clear that $a_j^{(m)}(\rho) = 0$, when ρ is replaced by λ_k: $a_j^{(m)}(\lambda_k) = 0$, $k = 1, \ldots, m$. Thus, (52) holds true.

Appendix B: Computation of the limits of Vandermonde determinants

Suppose a matrix $A(x) = [a_{kl}(x)]_{k,l=1}^m$ is given. The first derivative of the determinant of $A(x)$ is given by

$$(\det A(x))'_x = \mathrm{D}_x \left|a_{kl}(x)\right|_{k,l=1}^m = \sum_{j=0}^{m} \left|\mathrm{D}_x^{\delta_{jk}} a_{kl}(x)\right|_{k,l=1}^m = \sum_{j=0}^{m} \left|\mathrm{D}_x^{\delta_{jl}} a_{kl}(x)\right|_{k,l=1}^m, \tag{103}$$

where (D_x) is the differentiation operator with respect to x, $(D_x^0 = 1)$ is the identity operator and

$$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}, \quad j, k \in \mathbb{N}. \tag{104}$$

Suppose the determinant defined by (58) is given

$$\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m) = \left| v_{r+1,l}(\lambda_1, \ldots, \lambda_m) \right|_{r,l=0}^{m-1}, \tag{105}$$

where

$$v_{r+1,l} = \lambda_{r+1}^l, \quad l = 0, \ldots, m-2;\ r = 0, \ldots, m-1; \quad v_{r+1,m-1} = \lambda_{r+1}^j, \quad j \in \mathbb{Z}; \tag{106}$$

and $\lambda_k \in \mathbb{C}/\{0\}\ k = 1, \ldots, m; \lambda_k \neq \lambda_l,\ k \neq l.$

Given $\lambda_0 \neq 0$, the L'Hopital rule and (103) yield

$$\lim_{\lambda_1,\ldots,\lambda_m \to \lambda_0} \frac{\overline{V}_m^{(j)}(\lambda_1, \ldots, \lambda_m)}{V_m(\lambda_1, \ldots, \lambda_m)} = \lim_{\lambda_1,\ldots,\lambda_m \to \lambda_0} \frac{\begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{m-2} & \lambda_1^j \\ 1 & \lambda_2 & \ldots & \lambda_2^{m-2} & \lambda_2^j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda_m & \ldots & \lambda_m^{m-2} & \lambda_m^j \end{vmatrix}}{\begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_m & \ldots & \lambda_m^{m-1} \end{vmatrix}}$$

$$= \lim_{\lambda_1,\ldots,\lambda_m \to \lambda_0} \frac{\begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{m-2} & \lambda_1^j \\ \mathrm{D}_{\lambda_2} 1 & \mathrm{D}_{\lambda_2} \lambda_2 & \ldots & \mathrm{D}_{\lambda_2} \lambda_2^{m-2} & \mathrm{D}_{\lambda_2} \lambda_2^j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{D}_{\lambda_m}^{m+1} 1 & \mathrm{D}_{\lambda_m}^{m+1} \lambda_m & \ldots & \mathrm{D}_{\lambda_m}^{m+1} \lambda_m^{m-2} & \mathrm{D}_{\lambda_m}^{m+1} \lambda_m^j \end{vmatrix}}{\begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{m-1} \\ \mathrm{D}_{\lambda_2} 1 & \mathrm{D}_{\lambda_2} \lambda_2 & \ldots & \mathrm{D}_{\lambda_2} \lambda_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{D}_{\lambda_m}^{m+1} 1 & \mathrm{D}_{\lambda_m}^{m+1} \lambda_m & \ldots & \mathrm{D}_{\lambda_m}^{m+1} \lambda_m^{m-1} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_0^{m-2} & \lambda_0^j \\ 0 & 1 & \ldots & (m-2)\lambda_0^{m-3} & j\lambda_0^{j-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & j(j-1)(j-2)\ldots(j-m+2)\lambda_0^{j-m+1} \end{vmatrix}}{(m-1)!} = \binom{j}{m-1} \lambda_0^{j-m+1}. \tag{107}$$

Appendix C: Proof of Theorem 3

Let us construct an algebraic progression with coefficients

$$\overline{p}_j(\lambda_1, \ldots, \lambda_m) = \sum_{k=1}^{m} \nu_k \frac{\overline{V}_k^{(j)}(\lambda_1, \ldots, \lambda_k)}{V_k(\lambda_1, \ldots, \lambda_k)}, \tag{108}$$

where $m = 2, 3, \dots$ is fixed and $\overline{V}_1^{(j)}(\lambda_1) := \lambda_1$. Then (49) and (61) yield

$$\overline{p}_j(\lambda_1, \dots, \lambda_m) = \sum_{k=1}^{m} \nu_k \sum_{l=1}^{k} \frac{1}{\prod\limits_{\substack{r = 1, \dots, k \\ r \neq l}} (\lambda_l - \lambda_r)} \lambda_l^j$$

$$= \sum_{l=1}^{m} \left(\sum_{k=l}^{m} \frac{\nu_k}{\prod\limits_{\substack{r = 1, \dots, k \\ r \neq l}} (\lambda_l - \lambda_r)} \right) \lambda_l^j.$$

The highest order nonzero Hankel determinant of this sequence reads

$$\overline{d}_m^{(j)}(\lambda_1, \dots, \lambda_m) = \prod_{l=1}^{m} \left(\sum_{k=l}^{m} \frac{\mu_k}{\prod\limits_{\substack{r = 1, \dots, k \\ r \neq l}} (\lambda_l - \lambda_r)} \right) V_m^2(\lambda_1, \dots, \lambda_m) \prod_{k=1}^{m} \lambda_k^j$$

$$= \frac{\prod\limits_{l=1}^{m} \left(Q_l(\lambda_1, \dots, \lambda_m) + (-1)^{m-l} \mu_m \right)}{V_m^2(\lambda_1, \dots, \lambda_m)} V_m^2(\lambda_1, \dots, \lambda_m) \prod_{k=1}^{m} \lambda_k^j$$

$$= \prod_{l=1}^{m} \left(Q_l(\lambda_1, \dots, \lambda_m) + (-1)^{m-l} \mu_m \right) \prod_{k=1}^{m} \lambda_k^j \neq 0, \tag{109}$$

where $Q_l(\lambda_1, \dots, \lambda_m)$ are polynomials of $\lambda_1, \dots, \lambda_m$ satisfying equalities

$$\lim_{\lambda_1, \dots, \lambda_m \to \rho_0} Q_l(\lambda_1, \dots, \lambda_m) = 0, \quad l = 1, \dots, m-1; \tag{110}$$

and

$$Q_m(\lambda_1, \dots, \lambda_m) = 0. \tag{111}$$

Also, determinants $d_{m+n}^{(j)}(\lambda_1, \dots, \lambda_m) = 0$ for $n = 1, 2, \dots$ Therefore,

$$\text{Order}\left(p_j(\lambda_1, \dots, \lambda_m); j \in \mathbb{Z}\right) = m. \tag{112}$$

Let $\rho_0 \neq 0$. Then the following limit transitions hold true

$$\lim_{\lambda_1, \dots, \lambda_m \to \rho_0} \overline{d}_m^{(j)}(\lambda_1, \dots, \lambda_m) = (-1)^{\binom{m}{2}} \mu_m^m \rho_0^{mj} := d_m^{(j)}(\rho_0) \neq 0; \tag{113}$$

$$\lim_{\lambda_1, \dots, \lambda_m \to \rho_0} \overline{d}_{m+n}^{(j)}(\lambda_1, \dots, \lambda_m) := d_{m+n}^{(j)}(\rho_0) = 0, \quad n = 1, 2, \dots; \tag{114}$$

$$\lim_{\lambda_1, \dots, \lambda_m \to \rho_0} \overline{p}_j(\lambda_1, \dots, \lambda_m) = \sum_{k=1}^{m} \mu_k \binom{j}{k-1} \rho_0^{j-k+1} := p_j(\rho_0). \tag{115}$$

Thus,

$$\text{Order}\left(p_j\left(\rho_0\right);\ j \in \mathbb{Z}\right) = m. \tag{116}$$

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