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KINK SOLITARY SOLUTIONS TO A HEPATITIS C EVOLUTION MODEL

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Abstract. The standard nonlinear hepatitis C evolution model described in (Reluga et al. 2009) is considered in this paper. The generalized differential operator technique is used to construct analytical kink solitary solutions to the governing equations coupled with multiplicative and diffusive terms. Conditions for the existence of kink solitary solutions are derived. It appears that kink solitary solutions are either in a linear or in a hyperbolic relationship. Thus, a large perturbation in the population of hepatitis infected cells does not necessarily lead to a large change in uninfected cells. Computational experiments are used to illustrate the evolution of transient solitary solutions in the hepatitis C model.

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1. **Introduction.** Recent developments in computer hardware and software enable to use powerful symbolic computation techniques for the construction of nonlinear wave solutions to high-dimensional nonlinear evolution equations in mathematical physics. Solitary (or soliton) solutions represent solitary wave packets that do not change their shape when propagating at constant velocities [24]. Due to their unique properties, construction of solitary solutions is an important problem in nonlinear science [26, 4, 1].

A short overview of typical examples illustrating the discovery of solitary solutions in nonlinear evolution problems is presented below. The dynamic pressure of an irrotational solitary wave propagating at the surface of water over a flat bed is studied in [6]. Properties of bright and dark solitary solutions in strongly magnetized warm plasmas are considered in [5]. Solitary wave solutions to a system of coupled complex Newell-Segel-Whitehead equations are constructed in [9]. Gray/dark solitons in nonlocal nonlinear media are analytically studied using the symmetry reduction method in [8]. A closed-form analytical solution, including bright and dark solitons, to the driven nonlinear Schrödinger equation is constructed in [22].

Solitary solutions are often encountered in coupled differential equations. Next we mention several typical examples. Dark-bright soliton solutions to a coupled Schrödinger system with equal, repulsive cubic interactions are considered in [2]. In [23], exact bright one- and two-soliton solutions to a particular type of coherently coupled Schrödinger equations are constructed using the non-standard Hirota's bilinearization method. Three families of analytical solitary wave solutions of generalized coupled cubic-quintic Ginzburg-Landau equations are obtained in [27].

Even though kink solitary solutions are the simplest type of solitons, their construction and analysis is far from being trivial [21]. Kinks and bell-type soliton solutions to a differential equation describing the dynamics of microtubules are constructed in [28]. The interaction of kink-type solutions of harmonic map equations is studied in [7]. The kink solutions to the negative-order KdV equation are constructed using the Lax pair in [18]. Kink solutions to models of transport phenomena and mathematical biology are considered in [25].

The main objective of this paper is to seek kink solitary solutions in a hepatitis C virus infection model [20] that explicitly includes proliferation of infected and uninfected hepatocytes. The mathematical equations of the model are:

$$\begin{aligned}
\mathcal{T}'_{\widehat{t}} &= \widehat{s} + r_{\mathcal{T}}\mathcal{T}\left(1 - \frac{\mathcal{T}+\mathcal{I}}{\mathcal{T}_{max}}\right) - d_{\mathcal{T}}\mathcal{T} - (1-\eta)\,\beta\mathcal{V}\mathcal{T} + \widehat{q}\mathcal{I};\\
\mathcal{I}'_{\widehat{t}} &= r_{\mathcal{I}}\mathcal{I}\left(1 - \frac{\mathcal{T}+\mathcal{I}}{\mathcal{T}_{max}}\right) + (1-\eta)\,\beta\mathcal{V}\mathcal{T} - d_{\mathcal{I}}\mathcal{I} - \widehat{q}\mathcal{I};\\
\mathcal{V}'_{\widehat{t}} &= (1-\epsilon)\,p\mathcal{I} - c\mathcal{V},
\end{aligned} \tag{1}$$

where $\widehat{t}$ is time; $\mathcal{T}\left(\widehat{t}\right)$ represents uninfected hepatocytes; $\mathcal{I}\left(\widehat{t}\right)$ represents infected cells and $\mathcal{V}\left(\widehat{t}\right)$ represents free virus population. The parameters of (1) have the following meaning: p is the free virus production rate per infected cell; c is the immune virus clearance rate; $d_{\mathcal{T}}, d_{\mathcal{I}}$ are death rates for uninfected hepatocytes and infected cells respectively; $r_{\mathcal{T}}, r_{\mathcal{I}}$ are parameters of the logistic proliferation of $\mathcal{T}$ and $\mathcal{I}$ respectively; logistic proliferation happens only if $\mathcal{T} < \mathcal{T}_{max}$; β is the rate of infection per free virus per hepatocyte; parameters $\widehat{s}$ and $\widehat{q}$ represent the increase rate of uninfected hepatocytes through immigration and spontaneous cure by noncytolytic process respectively; finally the effect of antiviral treatment

reduces the infection rate by a fraction η and the viral production rate by a fraction ϵ. Ranges of parameters are given in [20].

The third equation of system (1) can be explicitly solved for V if patients are in a steady state before treatment. The introduction of dimensionless state variables x and y for uninfected and infected cells respectively reduces (1) to:

$$\begin{aligned} x'_\tau &= x\left(1-x-y\right)-\left(1-\theta\right)bxy+qy+s;\\ y'_\tau &= ry\left(1-x-y\right)+\left(1-\theta\right)bxy-dy-qy. \end{aligned} \tag{2}$$

Note that (2) can be rearranged in the general form:

$$\begin{aligned} x'_\tau &= a_0+a_1x+a_2x^2+a_3xy+a_4y; \quad x\Big|_{\tau=c}=u;\\ y'_\tau &= b_0+b_1y+b_2y^2+b_3xy+b_4x; \quad y\Big|_{\tau=c}=v, \end{aligned} \tag{3}$$

where $c,u,v,a_k,b_k\in\mathbb{R}, k=1,\dots,4$. System (3) is comprised of Riccati equations [19] coupled with both multiplicative and diffusive terms.

It appears that models (2) and (3) are natural extensions of the competing species evolution model with the Allee effect [12, 3]. Furthermore, similar evolution models are at the forefront of the analysis of population dynamics in many fields of research [11]. Thus, insight into the evolutionary dynamics of (3) would be valuable for understanding of transient processes in the hepatitis C model as well.

It has already been demonstrated that in the case $a_4=b_4=0$, system (3) does admit both kink and bright/dark solitary solutions [14, 16]. The aim of this paper is to construct kink solitary solutions to (3) when parameters a_4,b_4 are nonzero. Existence conditions in the space of the system parameters and explicit expressions of kink solutions are obtained using the generalized differential operator method.

2. **Preliminaries.**

2.1. **Kink solitary solutions.**

Kink solitary solutions have the following form [24, 17, 15]:

$$x\left(\tau;c,u,v\right)=\sigma\frac{\exp\left(\eta(\tau-c)\right)-x_1}{\exp\left(\eta(\tau-c)\right)-\tau_x}; \tag{4}$$

$$y\left(\tau;c,u,v\right)=\gamma\frac{\exp\left(\eta(\tau-c)\right)-y_1}{\exp\left(\eta(\tau-c)\right)-\tau_y}, \tag{5}$$

where $\sigma,\gamma,\tau_x,\tau_y,x_1,y_1$ are functions of initial conditions u,v; $\eta\in\mathbb{R}$. To simplify the analysis of (4), (5), the standard substitution

$$t:=\exp\left(\eta\tau\right),\quad \widehat{c}:=\exp\left(\eta c\right); \tag{6}$$

is used to transform (4), (5) into:

$$\widehat{x}\left(t;\widehat{c},u,v\right)=x\left(\frac{1}{\eta}\ln t;\frac{1}{\eta}\ln c,u,v\right)=\sigma\frac{t-\widehat{x}_1}{t-t_x}; \tag{7}$$

$$\widehat{y}\left(t;\widehat{c},u,v\right)=y\left(\frac{1}{\eta}\ln t;\frac{1}{\eta}\ln c,u,v\right)=\gamma\frac{t-\widehat{y}_1}{t-t_y}, \tag{8}$$

where $t_x, t_y, \widehat{x}_1, \widehat{y}_1$ depend on $\widehat{c}, u, v$. Furthermore (7) and (8) can be rearranged as:

$$\widehat{x} = \lambda_1 + \frac{\lambda_2}{1 - \rho_x (t - \widehat{c})}; \tag{9}$$

$$\widehat{y} = \mu_1 + \frac{\mu_2}{1 - \rho_y (t - \widehat{c})}, \tag{10}$$

with $\lambda_k = \lambda_k (\widehat{c}, u, v), \mu_k = \mu_k (\widehat{c}, u, v), \rho_x = \rho_x (\widehat{c}, u, v), \rho_y = \rho_y (\widehat{c}, u, v)$.

In the remainder of this paper, kink solutions written in the form (9),(10) will be considered.

2.2. **Operator expression of solutions to systems of nonlinear ODEs.** Let P, Q be trivariate analytic functions. Consider the following system of differential equations:

$$\begin{aligned} \widehat{x}'_t &= P (t, \widehat{x}, \widehat{y}); \quad \widehat{x}\Big|_{t=\widehat{c}} = u; \\ \widehat{y}'_t &= Q (t, \widehat{x}, \widehat{y}); \quad \widehat{y}\Big|_{t=\widehat{c}} = v. \end{aligned} \tag{11}$$

The generalized differential operator with respect to system (11) is defined as [14]:

$$\mathbf{D}_{\widehat{c}uv} := \mathbf{D}_{\widehat{c}} + P (\widehat{c}, u, v) \mathbf{D}_u + Q (\widehat{c}, u, v) \mathbf{D}_v, \tag{12}$$

where $\mathbf{D}_{\widehat{c}}, \mathbf{D}_u, \mathbf{D}_v$ are partial differentiation operators with respect to the indexed variables. Note that standard properties of differential operators [13] do hold for (12).

The general solution to Eq.(11), [14, 13], takes the following form:

$$\widehat{x} = \sum_{j=0}^{+\infty} \frac{(t - \widehat{c})^j}{j!} \mathbf{D}^j_{\widehat{c}uv} u; \quad \widehat{y} = \sum_{j=0}^{+\infty} \frac{(t - \widehat{c})^j}{j!} \mathbf{D}^j_{\widehat{c}uv} v. \tag{13}$$

Note that $\mathbf{D}^0_{\widehat{c}uv} = \mathbf{I}$, where $\mathbf{I}$ is the identity operator.

3. **Existence conditions for kink solutions to (11).** Let

$$p_j := \mathbf{D}^j_{\widehat{c}uv} u; \quad q_j = \mathbf{D}^j_{\widehat{c}uv} v, \quad j = 0, 1, \ldots. \tag{14}$$

Note that the functions $p_j = p_j (\widehat{c}, u, v); q_j = q_j (\widehat{c}, u, v)$ satisfy recurrence relations $p_{j+1} = \mathbf{D}_{\widehat{c}uv} p_j;\ q_{j+1} = \mathbf{D}_{\widehat{c}uv} q_j$. Furthermore, (13) and (14) yield:

$$\widehat{x} = \sum_{j=0}^{+\infty} \frac{(t - \widehat{c})^j}{j!} p_j; \tag{15}$$

$$\widehat{y} = \sum_{j=0}^{+\infty} \frac{(t - \widehat{c})^j}{j!} q_j. \tag{16}$$

If (11) admits the solution (9),(10) then (15) and (9) must be equal, so that:

$$\sum_{j=0}^{+\infty} \frac{(t - \widehat{c})^j}{j!} p_j = \lambda_1 + \frac{\lambda_2}{1 - \rho_x (t - \widehat{c})}. \tag{17}$$

Expanding the right side of (17) results in:

$$\sum_{j=0}^{+\infty} \frac{(t-\widehat{c})^j}{j!} p_j = \lambda_1 + \lambda_2 + \sum_{j=1}^{+\infty} \frac{(t-\widehat{c})^j}{j!} \left(j! \lambda_2 \rho_x^j \right). \tag{18}$$

Equality (18) together with $p_0 = u$ yields:

$$u = \lambda_1 + \lambda_2; \tag{19}$$

$$p_j = j! \lambda_2 \rho_x^j, \ j = 1, 2, \ldots. \tag{20}$$

Analogously, (16) and (10) yield:

$$v = \mu_1 + \mu_2; \tag{21}$$

$$q_j = j! \mu_2 \rho_y^j, \ j = 1, 2, \ldots. \tag{22}$$

Theorem 3.1. *System (11) admits kink solitary solutions (9) and (10) if and only if the following conditions hold true for all values of $\widehat{c}, u, v$:*

$$\lambda_2 = \frac{p_1}{\rho_x}, \quad \rho_x \neq 0; \qquad \mu_2 = \frac{q_1}{\rho_y}, \quad \rho_y \neq 0; \tag{23}$$

$$\mathbf{D}_{\widehat{c}uv} \rho_x = \rho_x^2; \qquad \mathbf{D}_{\widehat{c}uv} \rho_y = \rho_y^2; \tag{24}$$

$$\mathbf{D}_{\widehat{c}uv} \lambda_2 = \lambda_2 \rho_x; \qquad \mathbf{D}_{\widehat{c}uv} \mu_2 = \mu_2 \rho_y. \tag{25}$$

Proof. Derivations presented in Section 3 yield that system (11) admits kink solitary solutions if and only if (20) and (22) hold true. Thus it will be proven that conditions (23)–(25) are necessary and sufficient for (20) and (22) to hold true.
Necessity. Let (20) hold true. Taking $j = 1, 2$ results in:

$$p_1 = \lambda_2 \rho_x; \tag{26}$$

$$p_2 = 2\lambda_2 \rho_x^2. \tag{27}$$

Solving (26) for λ_2 yields (23). Solution to (27) for ρ_x together with (23) results in:

$$\rho_x = \frac{p_2}{2p_1}, \quad p_1 \neq 0. \tag{28}$$

Equation (28) results in (24):

$$\mathbf{D}_{\widehat{c}uv} \rho_x = \frac{2p_1 \left(\mathbf{D}_{\widehat{c}uv} p_2 \right) - 2p_2 \left(\mathbf{D}_{\widehat{c}uv} p_1 \right)}{4p_1^2} = \frac{12\lambda_2^2 \rho_x^4 - 8\lambda_2^2 \rho_x^4}{4\lambda_2^2 \rho_x^2} = \rho_x^2. \tag{29}$$

Analogously, (23) yields (25):

$$\mathbf{D}_{\widehat{c}uv} \lambda_2 = \mathbf{D}_{\widehat{c}uv} \frac{p_1}{\rho_x} = \frac{\rho_x \left(\mathbf{D}_{\widehat{c}uv} p_1 \right) - p_1 \left(\mathbf{D}_{\widehat{c}uv} \rho_x \right)}{\rho_x^2} = \frac{2\lambda_2 \rho_x^3 - \lambda_2 \rho_x^3}{\rho_x^2} = \lambda_2 \rho_x. \tag{30}$$

Sufficiency. Condition (23) yields:

$$p_1 = \lambda_2 \rho_x. \tag{31}$$

Applying the operator $\mathbf{D}_{\widehat{c}uv}$ to (31) and using (24) and (25) yields:

$$p_2 = \mathbf{D}_{\widehat{c}uv} \left(\lambda_2 \rho_x \right) = \rho_x \left(\mathbf{D}_{\widehat{c}uv} \lambda_2 \right) + \lambda_2 \left(\mathbf{D}_{\widehat{c}uv} \rho_x \right) = 2\lambda_2 \rho_x^2. \tag{32}$$

Thus, continuing by induction, if $p_j = j! \lambda_2 \rho_x^j$, then:

$$\begin{aligned} p_{j+1} &= \mathbf{D}_{\widehat{c}uv} \left(j! \lambda_2 \rho_x^j \right) = j! \left(\rho_x^j \left(\mathbf{D}_{\widehat{c}uv} \lambda_2 \right) + j \lambda_2 \rho_x^{j-1} \left(\mathbf{D}_{\widehat{c}uv} \rho_x \right) \right) \\ &= j! \left(\lambda_2 \rho_x^{j+1} + j \lambda_2 \rho_x^{j+1} \right) = (j+1)! \lambda_2 \rho_x^{j+1}, \end{aligned} \tag{33}$$

which finishes the proof. □

Corollary 1. *Equations (19) and (28) yield:*

$$\lambda_1 = u - \frac{2p_1^2}{p_2}, \tag{34}$$

furthermore:

$$\mathbf{D}_{\widehat{c}uv}\lambda_1 = \mathbf{D}_{\widehat{c}uv}u - \mathbf{D}_{\widehat{c}uv}\lambda_2 = 0. \tag{35}$$

Analogous relations hold for μ_1.

Corollary 2. *Let (20) hold true. Then:*

$$2p_1p_3 - 3p_2^2 = 0. \tag{36}$$

Equivalently, (22) results in:

$$2q_1q_3 - 3q_2^2 = 0. \tag{37}$$

4. **Construction of kink solutions to (3).** Results of Theorem 3.1 are used to determine existence conditions of kink solutions to (3) in this section. Furthermore, explicit expressions of kink solutions are given in terms of the system's parameters.

4.1. **Transformation of (3).** Applying transformation (6) to (3) results in the following system:

$$\begin{aligned} \eta t\widehat{x}_t' &= a_0 + a_1\widehat{x} + a_2\widehat{x}^2 + a_3\widehat{x}\widehat{y} + a_4\widehat{y}; \\ \eta t\widehat{y}_t' &= b_0 + b_1\widehat{y} + b_2\widehat{y}^2 + b_3\widehat{x}\widehat{y} + b_4\widehat{x}. \end{aligned} \tag{38}$$

System (38) is subject to initial conditions:

$$\widehat{x}\Big|_{t=\widehat{c}} = u; \quad \widehat{y}\Big|_{t=\widehat{c}} = v. \tag{39}$$

4.2. **Derivation of existence conditions of kink solutions to (3).** The generalized differential operator $\mathbf{D}_{\widehat{c}uv}$ with respect to (38) reads:

$$\begin{aligned} \mathbf{D}_{\widehat{c}uv} :=& \mathbf{D}_{\widehat{c}} + \frac{1}{\eta\widehat{c}}\Big(a_0 + a_1u + a_2u^2 + a_3uv + a_4v\Big)\mathbf{D}_u \\ &+ \frac{1}{\eta\widehat{c}}\Big(b_0 + b_1v + b_2v^2 + b_3uv + b_4u\Big)\mathbf{D}_v. \end{aligned} \tag{40}$$

The conditions of Theorem 3.1 can only be satisfied for special values of parameters $\eta; a_0, \ldots, a_4; b_0, \ldots, b_4$. Derivation of these parameter values is given in the following subsection.

4.2.1. *Computation of parameter η.* Coefficients $p_j, q_j; j = 1, 2, 3$ are computed using (14) and (40). Note that to satisfy Theorem 3.1, η must also satisfy Corollary 2. Equations (36), (37) result in the following system:

$$2p_1p_3 - 3p_2^2 = 0; \tag{41}$$

$$2q_1q_3 - 3q_2^2 = 0. \tag{42}$$

Values of η can be computed from (41) and (42) using computer algebra. First it can be noted that (41) and (42) have the following structure:

$$A_p(u,v)\eta^2 + B_p(u,v) = 0; \tag{43}$$

$$A_q(u,v)\eta^2 + B_q(u,v) = 0, \tag{44}$$

where A_p, A_q, B_p, B_q are known functions of u, v. Equation (43) yields:

$$\eta^2 = -\frac{B_p(u,v)}{A_p(u,v)}. \tag{45}$$

If the parameter η is not a function of $\widehat{c}, u, v$, then the kink solution definition holds true. Furthermore, if η does not depend on $\widehat{c}, u, v$ sufficient conditions of Theorem 3.1 hold true.

Long division of (45) results in:

$$\eta^2 = S_p(u,v) + \frac{R_p(u,v)}{A_p(u,v)}. \tag{46}$$

Analogously, (44) yields:

$$\eta^2 = S_q(u,v) + \frac{R_q(u,v)}{A_q(u,v)}. \tag{47}$$

Parameter η does not depend on u, v only if $a_0, \ldots, a_4; b_0, \ldots, b_4$ are chosen in such a way that $S_p = S_q$ are independent of u, v and

$$R_p = R_q = 0. \tag{48}$$

4.2.2. *Solution of (48).* Remainder R_p has the following structure:

$$R_p = \frac{1}{a_2^2}\left(u^3 \sum_{j=0}^{3} c_{3j} v^j + u^2 \sum_{j=0}^{4} c_{2j} v^j + u \sum_{j=0}^{4} c_{1j} v^j + \sum_{j=0}^{4} c_{0j} v^j\right), \tag{49}$$

where coefficients c_{kj} depend only on $a_0, \ldots, a_4; b_0, \ldots, b_4$. Equations (48) and (49) result in the following system of 19 algebraic equations:

$$\begin{aligned} &c_{3j} = 0, \quad j = 0, \ldots, 3; \\ &c_{kj} = 0, \quad k = 0, 1, 2, \quad j = 0, \ldots, 4. \end{aligned} \tag{50}$$

Solution to (50) reads:

$$a_0 = \frac{a_4\left(a_1 a_3 - a_2 a_4\right)}{a_3^2}; \tag{51}$$

$$b_0 = -\frac{a_1^2 a_3^2 b_3 - a_1 a_2 a_3^2 b_1 - 2 a_1 a_2 a_3 a_4 b_3 + a_2^2 a_3 a_4 b_1 + a_2^2 a_4^2 b_3}{a_2 a_3^3}; \tag{52}$$

$$b_2 = \frac{a_3 b_3}{a_2}; \tag{53}$$

$$b_4 = -\frac{a_1 a_3 b_3 - a_2 a_3 b_1 - a_2 a_4 b_3}{a_3^2}. \tag{54}$$

Inserting (51)–(54) into R_q results in $R_q = 0$, thus (48) is satisfied.

4.2.3. *Final solution for η.* Inserting (51)–(54) into S_p, S_q yields:

$$\begin{aligned} S_p = {} & \frac{2 a_3 b_3 \left(a_2 - b_3\right)}{a_2} uv + \frac{a_3^2\left(a_2^2 - b_3^2\right)}{a_2^2} v^2 - \frac{2\left(a_2 - b_3\right)\left(a_1 a_3 b_3 - a_2 a_3 b_1 - a_2 a_4 b_3\right)}{a_2 a_3} u \\ & + \frac{2\left(a_1 a_2 a_3 + a_1 a_3 b_3 - 2 a_2^2 a_4 - a_2 a_4 b_3\right)\left(a_2 - b_3\right)}{a_2^2} v \\ & + \frac{1}{a_2^2 a_3^2}\Big(a_1^2 a_2^2 a_3^2 + 3 a_1^2 a_3^2 b_3^2 - 4 a_1 a_2^3 a_3 a_4 - 2 a_1 a_2^2 a_3 a_4 b_3 \end{aligned}$$

$$
\begin{aligned}
&- 4a_1a_2a_3^2b_1b_3 - 6a_1a_2a_3a_4b_3^2 + 4a_2^4a_4^2 + 2a_2^3a_3a_4b_1 + 2a_2^3a_4^2b_3 + a_2^2a_3^2b_1^2 \\
&+ 4a_2^2a_3a_4b_1b_3 + 3a_2^2a_4^2b_3^2\Big);
\end{aligned} \tag{55}
$$

and

$$
\begin{aligned}
S_q = &-2a_3\left(a_2-b_3\right)uv + \left(b_3^2-a_2^2\right)u^2 - \frac{2\left(a_2-b_3\right)}{a_2a_3} \\
&\times\left(-a_4a_2^2 + \left(\left(a_1-b_1\right)a_3 - 2a_4b_3\right)a_2 + 2a_1a_3b_3\right)u - 2a_4\left(a_2-b_3\right)v \\
&+ \frac{1}{a_3^2a_2^2}\Big(4a_1^2a_3^2b_3^2 - 2a_1a_2^3a_3a_4 - 2a_1a_2^2a_3a_4b_3 - 4a_1a_2a_3^2b_1b_3 \\
&- 8a_1a_2a_3a_4b_3^2 + 3a_2^4a_4^2 + 2a_2^3a_3a_4b_1 + 2a_2^3a_4^2b_3 + a_2^2a_3^2b_1^2 + 4a_2^2a_3a_4b_1b_3 \\
&+ 4a_2^2a_4^2b_3^2\Big).
\end{aligned} \tag{56}
$$

From (55) and (56) it follows that S_p, S_q are equal and do not depend on u, v in the following two cases:

Case 1. If

$$
b_3 = a_2, \tag{57}
$$

then

$$
\eta^2 = S_p = S_q = \left(\frac{2a_1a_3 - 3a_2a_4 - a_3b_1}{a_3}\right)^2. \tag{58}
$$

Conditions (51)–(54) read:

$$
a_0 = \frac{a_4\left(a_1a_3 - a_2a_4\right)}{a_3^2}; \tag{59}
$$

$$
b_0 = -\frac{\left(a_1a_3 - a_2a_4\right)\left(a_1a_3 - a_2a_4 - a_3b_1\right)}{a_3^3}; \tag{60}
$$

$$
b_2 = a_3; \tag{61}
$$

$$
b_4 = \frac{a_2\left(a_2a_4 + a_3b_1 - a_1a_3\right)}{a_3^2}. \tag{62}
$$

Note that coefficients a_k and b_k can be interchanged in (57)–(62) because of symmetry.

Case 2. If $b_3 = -a_2$ then (55) and (56) read:

$$
\begin{aligned}
\eta^2 = S_p = S_q = &-4a_2a_3uv + \frac{4\left(\left(a_1+b_1\right)a_3 - a_2a_4\right)a_2}{a_3}u - 4a_2a_4v \\
&+ \frac{4\left(a_1+\frac{b_1}{2}\right)^2a_3^2 - 8a_4a_2\left(a_1+\frac{b_1}{2}\right)a_3 + 5a_4^2a_2^2}{a_3^2}.
\end{aligned} \tag{63}
$$

From (63) it follows that S_p, S_q is independent of u, v only if:

$$
b_3 = a_2 = 0. \tag{64}
$$

Condition (64) transforms (63) into:

$$
\eta^2 = \left(b_1 + 2a_1\right)^2. \tag{65}
$$

Conditions (51)–(54) read:

$$a_0 = \frac{a_1 a_4}{a_3}; \tag{66}$$

$$b_0 = \frac{a_1 (a_1 + b_1)}{a_3}; \tag{67}$$

$$b_2 = -a_3; \tag{68}$$

$$b_4 = 0. \tag{69}$$

4.3. **Kink solutions in linear relationship.**

Theorem 4.1. *System (38) admits kink solitary solutions for all initial conditions $\widehat{c}, u, v$ if the parameters $\eta; a_0, \ldots, a_4; b_0, \ldots, b_4$ satisfy the following relations:*

$$\eta = \pm \frac{2a_1 a_3 - 3a_2 a_4 - a_3 b_1}{a_3}; \tag{70}$$

$$b_2 = a_3; \quad a_2 = b_3; \tag{71}$$

$$b_4 (b_1 b_3 - b_2 b_4) = b_0 b_3^2; \tag{72}$$

$$a_2 (a_1 a_3 - a_2 a_4) = b_2 (b_1 b_3 - b_2 b_4); \tag{73}$$

$$a_0 a_3 b_4 = b_0 b_3 a_4. \tag{74}$$

Proof. Let (57), (59)–(62) hold true. Equation (70) follows from (58).

1. Let us consider the (+) branch of (70).
   Condition (28) yields:

$$\rho_x = \rho_y = \frac{a_3^2 v + a_3 b_3 u - a_1 a_3 + a_3 b_1 + 2a_4 b_3}{\widehat{c}\,(2a_1 a_3 - a_3 b_1 - 3a_4 b_3)}. \tag{75}$$

   Applying operator (40) to (75) results in (24).
   Condition (23) yields values of λ_2, μ_2:

$$\lambda_2 = \frac{(a_3 u + a_4)\left(v a_3^2 + (b_3 u + a_1) a_3 - a_4 b_3\right)}{a_3\left(v a_3^2 + (b_3 u - a_1 + b_1) a_3 + 2a_4 b_3\right)}; \tag{76}$$

$$\mu_2 = \frac{\left(v a_3^2 + (b_1 - a_1) a_3 + a_4 b_3\right)\left(v a_3^2 + (b_3 u + a_1) a_3 - a_4 b_3\right)}{a_3^2\left(v a_3^2 + (b_3 u - a_1 + b_1) a_3 + 2a_4 b_3\right)}. \tag{77}$$

   From (76), (77) and (40) it follows that (25) holds true. Thus, (38) admits kink solutions.
2. For the (−) branch of (70) steps of the proof are repeated, yielding:

$$\rho_x = \rho_y = \frac{a_4 b_3 - a_1 a_3 - a_3 b_3 u - a_3^2 v}{\widehat{c}\,(2a_1 a_3 - a_3 b_1 - 3a_4 b_3)}; \tag{78}$$

$$\lambda_2 = \frac{1}{a_3}(a_3 u + a_4); \quad \mu_2 = \frac{1}{a_3^2}\left(a_4 b_3 + a_3 b_1 - a_1 a_3 + a_3^2 v\right). \tag{79}$$

   Conditions (24) and (25) hold true for (78) and (79).

It can be shown that symmetric conditions (71)–(74) are equivalent to (57), (59)–(62). Relations (59) and (62) can be rewritten as:

$$a_1 a_3 - a_2 a_4 = \frac{a_0 a_3^2}{a_4}; \tag{80}$$

$$a_1 a_3 - a_2 a_4 = a_3 b_1 - \frac{b_4 a_3^2}{a_2}. \tag{81}$$

Inserting (80) and (81) into (60) yields:

$$b_0 = \frac{a_0 b_4 a_3}{a_2 a_4}, \tag{82}$$

which results in (74).

Equation (81) together with (71) yields (73), which can be rewritten as:

$$b_1 b_3 - b_2 b_4 = \frac{a_2}{b_2}\left(a_1 a_3 - a_2 a_4\right). \tag{83}$$

Using (80) and (71) transforms (83) into:

$$b_1 b_3 - b_2 b_4 = \frac{b_0 b_3^2}{b_4}, \tag{84}$$

that results in (72). □

Corollary 3. *It follows from subsection 4.1 that system (3) admits kink solutions of the form (4), (5) when conditions of Theorem 4.1 hold true. Parameters of (4), (5) read:*

$$\sigma = \lambda_1; \quad \gamma = \mu_1; \tag{85}$$

$$\tau_x = \tau_y = \frac{1 + \widehat{c}\rho_x}{\widehat{c}\rho_x}; \tag{86}$$

$$x_1 = \frac{1}{\widehat{c}\rho_x \lambda_1}\left(\lambda_1\left(1 + \widehat{c}\rho_x\right) + \lambda_2\right); \tag{87}$$

$$y_1 = \frac{1}{\widehat{c}\rho_x \mu_1}\left(\mu_1\left(1 + \widehat{c}\rho_x\right) + \mu_2\right), \tag{88}$$

where ρ_x, λ_2, μ_2 are given by (75), (76) and (77) or (78) and (79) depending on the sign of η. Parameters λ_1, μ_1 are defined in Corollary 1.

Note that $\widehat{c}\rho_x = \widehat{c}\rho_y$ does not depend on $\widehat{c}$, thus (85)–(88) depend only on initial conditions u, v. Also note that cases with positive and negative signs of η are interchangeable, because:

$$x = \sigma\frac{\exp\left(\eta(\tau - c)\right) - x_1}{\exp\left(\eta(\tau - c)\right) - \tau_x} = \frac{\sigma x_1}{\tau_x}\frac{\exp\left(-\eta\left(\tau - c\right)\right) - \frac{1}{x_1}}{\exp\left(-\eta\left(\tau - c\right)\right) - \frac{1}{\tau_x}}. \tag{89}$$

Analogous rearrangements also hold true for y. Thus in further computations only one sign of η can be considered.

Corollary 4. *Let conditions of Theorem 4.1 hold true. Kink solutions to (3) satisfy the following linear relationship:*

$$Ax(\tau) + By(\tau) = 1, \tag{90}$$

where

$$A = \frac{a_1 a_3 - a_3 b_1 - a_4 b_3 - a_3^2 v}{\left(a_1 a_3 - a_3 b_1 - a_4 b_3\right) u + a_3 a_4 v}; \tag{91}$$

$$B = \frac{a_3\left(a_3 u + a_4\right)}{\left(a_1 a_3 - a_3 b_1 - a_4 b_3\right) u + a_3 a_4 v}. \tag{92}$$

Proof. Equation (4), (5) yields:

$$Ax(\tau) + By(\tau) = \frac{\left(A\sigma + B\gamma\right)\exp\left(\eta\left(\tau - c\right)\right) - \left(A\sigma x_1 + B\gamma y_1\right)}{\exp\left(\eta\left(\tau - c\right)\right) - \tau_x}. \tag{93}$$

Kink solutions are in linear relationship (90) if:

$$\begin{aligned} &A\sigma + B\gamma = 1; \\ &A\sigma x_1 + B\gamma y_1 = \tau_x. \end{aligned} \tag{94}$$

Solution of (94) together with (85)–(88) result in (91) and (92). □

Corollary 4 demonstrates that all phase plane trajectories of the system (3) are straight lines if the conditions of Theorem 4.1 hold true.

4.4. **Kink solutions in hyperbolic relationship.**

Theorem 4.2. *System (38) admits kink solitary solutions for all initial conditions $\widehat{c}, u, v$ if (64), (66)–(69) hold true and*

$$\eta = -b_1 - 2a_1. \tag{95}$$

Proof. Let (64), (66)–(69) hold true. It follows from (65) that:

$$\eta = \pm\left(b_1 + 2a_1\right). \tag{96}$$

1. Let us consider the $(+)$ branch of (96). It can be obtained that $p_j = 0, j = 2, 3, \ldots$, thus by Theorem 3.1 $\widehat{x}$ cannot be a solitary solution.
2. Considering the $(-)$ branch of (96) yields the following parameters:

$$\rho_x = -\frac{1}{\widehat{c}}; \quad \rho_y = \frac{a_3 v - a_1 - b_1}{\widehat{c}\left(b_1 + 2a_1\right)}; \tag{97}$$

$$\lambda_2 = \frac{\left(a_3 v + a_1\right)\left(a_3 u + a_4\right)}{a_3\left(b_1 + 2a_1\right)}; \quad \mu_2 = \frac{a_3 v + a_1}{a_3}. \tag{98}$$

The conditions of Theorem 3.1 are satisfied for (97), (98), which finishes the proof.

□

Remark 1. Note that analytical non-kink solutions do exist for $\eta = b_1 + 2a_1$. Equation (15) yields that $\widehat{x}$ is a linear functions of t:

$$\widehat{x} = \frac{\left(a_3 u + a_4\right)\left(\left(a_3 v + a_1\right)t - \left(a_3 v - a_1 - b_1\right)\widehat{c}\right)}{\widehat{c} a_3\left(b_1 + 2a_1\right)} - \frac{a_4}{a_3}, \tag{99}$$

while $\widehat{y}$ is a kink solution with the following parameters:

$$\rho_y = -\frac{a_3 v + a_1}{\widehat{c}\left(b_1 + 2a_1\right)}, \quad \mu_2 = \frac{a_3 v - a_1 - b_1}{a_3}, \quad \mu_1 = \frac{a_1 + b_1}{a_3}. \tag{100}$$

Corollary 5. *When conditions of Theorem 4.2 hold true, system (3) admits kink solutions of the form (4), (5) with the following parameters:*

$$\sigma = \lambda_1; \quad \gamma = \mu_1; \tag{101}$$

$$\tau_x = 0; \tag{102}$$

$$\tau_y = \frac{1 + \widehat{c}\rho_y}{\widehat{c}\rho_y}; \tag{103}$$

$$x_1 = \frac{1}{\widehat{c}\rho_x \lambda_1}\left(\lambda_1\left(1 + \widehat{c}\rho_x\right) + \lambda_2\right); \tag{104}$$

$$y_1 = \frac{1}{\widehat{c}\rho_x \mu_1}\left(\mu_1\left(1 + \widehat{c}\rho_x\right) + \mu_2\right), \tag{105}$$

where $\rho_x, \rho_y, \lambda_2, \mu_2$ are given in (97) and (98) and λ_1, μ_1 are computed as in Corollary 1.

Corollary 6. *If conditions of Theorem 4.2 hold true, then kink solutions x, y to (3) satisfy the following hyperbolic relationship:*

$$x(\tau)\,y(\tau) - \frac{a_1+b_1}{a_3}x(\tau) + \frac{a_4}{a_3}y(\tau) = uv - \frac{a_1+b_1}{a_3}u + \frac{a_4}{a_3}v. \tag{106}$$

Proof. Let B, C be unknown coefficients. Equations (101)–(105) together with (4), (5) result in:

$$\begin{aligned} &x(\tau)\,y(\tau) - Bx(\tau) - Cy(\tau) - uv + Bu + Cv \\ &= \frac{\left(1-\exp\left(\eta(t-c)\right)\right)(a_3v+a_1)\left((a_1+b_1-a_3v)\Xi(B,C)\exp\left(\eta(t-c)\right)+(a_3v+a_1)(a_3u+a_4)\Theta(B,C)\right)}{a_3^2\eta\exp\left(\eta(t-c)\right)\left((a_1+b_1-a_3v)\left(\exp\left(\eta(t-c)\right)+a_3v+a_1\right)\right)}, \end{aligned} \tag{107}$$

where

$$\Xi(B,C) = -Bua_3^2 - \left((2C+u)\,a_1 + (C+u)\,b_1 + a_4B\right)a_3 + a_1a_4; \tag{108}$$

$$\Theta(B,C) = Ba_3 + a_1 + b_1. \tag{109}$$

Solving $\Xi = 0, \Theta = 0$ for B, C finishes the proof. □

Corollary 6 shows that all phase plane trajectories of system (3) are hyperbolas when conditions of Theorem 4.2 hold true.

5. Computational experiments.

5.1. Kink solutions in linear relationship.

5.1.1. *Equilibria.* Let the conditions of Theorem 4.1 hold true. Equilibrium points of system (3) are given by:

$$x_*^{(1)} = -\frac{a_4}{a_3}, \quad y_*^{(1)} = \frac{a_1a_3 - a_2a_4 - a_3b_1}{a_3^2} = -\frac{b_4}{b_3}; \tag{110}$$

$$b_2x_*^{(2)} + a_2y_*^{(2)} = -\frac{a_0a_3}{a_4}. \tag{111}$$

Note that (111) describes infinitely many equilibria that lie on a straight line.

5.1.2. *Construction of kink solutions.* Let us consider the following system:

$$\begin{aligned} x'_\tau &= -\frac{3}{8} + 2x - 2x^2 + 4xy - 3y; \\ y'_\tau &= -\frac{3}{16} - y + 4y^2 - 2xy + \frac{3}{4}x; \\ x\Big|_{\tau=c} &= u, \quad y\Big|_{\tau=c} = v. \end{aligned} \tag{112}$$

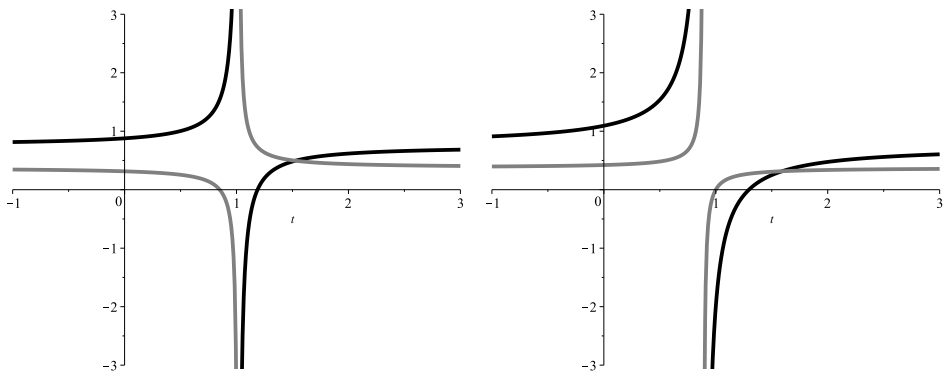
Using transformation (6) on (112) results in:

$$\begin{aligned} \eta t\widehat{x}'_t &= -\frac{3}{8} + 2\widehat{x} - 2\widehat{x}^2 + 4\widehat{x}\widehat{y} - 3\widehat{y}; \\ \eta t\widehat{y}'_t &= -\frac{3}{16} - \widehat{y} + 4\widehat{y}^2 - 2\widehat{x}\widehat{y} + \frac{3}{4}\widehat{x}; \\ \widehat{x}\Big|_{t=\widehat{c}} &= u, \quad \widehat{y}\Big|_{t=\widehat{c}} = v. \end{aligned} \tag{113}$$

Coefficients of (112) satisfy (71)–(74). Equation (70) yields:

$$\eta = \pm\frac{1}{2}. \tag{114}$$

By (89), it is enough to consider $\eta = -\frac{1}{2}$ to obtain all solutions to (112).



(a) (b)

FIGURE 1. Kink solutions $\widehat{x}, \widehat{y}$ to (112) with $\widehat{c} = 1$. The black line denotes $\widehat{x}(t)$; the gray line denotes $\widehat{y}(t)$. In (a), $u = 10, v = -4$; in (b), $u = -2, v = 0$.

By Theorem 4.1, the parameters of kink solutions to (113) read:

$$\lambda_1 = \frac{3}{4}; \quad \lambda_2 = u - \frac{3}{4}; \tag{115}$$

$$\mu_1 = \frac{3}{8}; \quad \mu_2 = v - \frac{3}{8}; \tag{116}$$

$$\rho_x = \rho_y = \frac{1}{\widehat{c}}\left(4u - 8v - 1\right). \tag{117}$$

Kink solutions $\widehat{x}, \widehat{y}$ to (113) are depicted in Fig. 1.

Parameters of kink solutions of the form (4), (5) read:

$$\sigma = \frac{3}{4}; \quad \gamma = \frac{3}{8}; \quad \tau_x = \tau_y = \frac{4u - 8v}{4u - 8v - 1} \tag{118}$$

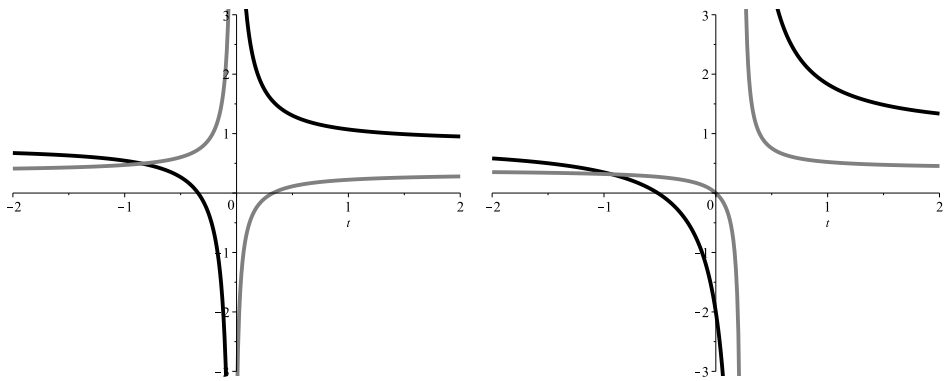
$$x_1 = \frac{4\left(16u - 24v - 3\right)}{12\left(4u - 8v - 1\right)}; \quad y_1 = \frac{8\left(12u - 16v - 3\right)}{24\left(4u - 8v - 1\right)}. \tag{119}$$

Kink solutions x, y with parameters (118), (119) are depicted in Fig. 2. Note that Fig. 2 (a) and (b) correspond to Fig. 1 (a) and (b), since $c = \frac{1}{\eta}\ln\widehat{c}$.

Note that by Corollary 4, kink solutions do satisfy the following linear relationship:

$$\frac{3 - 8v}{3u - 6v}x + \frac{8u - 6v}{3u - 6v}y = 1. \tag{120}$$

The phase plane of (112) is depicted in Fig. 3. Because (120) holds true, all phase plane trajectories of (112) are straight lines. Note that the equilibrium point defined by (110) is an unstable node. The equilibrium line defined by (111) is attractive on the half-plane that does not contain the unstable node. On the half plane that does contain the unstable node, the line is repulsive.



(a) (b)

FIGURE 2. Kink solutions x, y to (112) with $c = 0$. The black line denotes $x(\tau)$; the gray line denotes $y(\tau)$. In (a), $u = 10, v = -4$; in (b), $u = -2, v = 0$.

Because kink solutions are in linear relationship, a perturbation in infected cell population y leads to a proportional perturbation in uninfected cell population x. In Fig. 3, as solutions evolve from point A to B, y increases by 0.46 while x decreases by 1.09.

5.1.3. *Control of system parameters.* Consider the following system:

$$\begin{aligned} x'_\tau &= -\frac{3}{8} + a_1 x + b_3 x^2 + 4xy + a_4 y; \\ y'_\tau &= -\frac{3}{16} + b_1 y + 4y^2 + b_3 xy + \frac{3}{4}x; \\ x\Big|_{\tau=c} &= u, \quad y\Big|_{\tau=c} = v, \end{aligned} \tag{121}$$

where $a_1, a_3, a_4, b_1 \in \mathbb{R}$. Condition (71) of Theorem 4.1 is satisfied. System (121) admits kink solitary solutions if (72)–(74) hold true:

$$\frac{3}{4} b_1 b_3 - \frac{9}{4} = -\frac{3}{16} b_3^2; \tag{122}$$

$$b_3 (4a_1 - b_3 a_4) = 4 b_1 b_3 - 12; \tag{123}$$

$$-\frac{9}{8} = -\frac{3}{16} b_3 a_4. \tag{124}$$

Since (122)–(124) contain parameter b_3, this parameter can be designated as the control parameter for the existence of kink solitary solutions. Solving (122)–(124) with respect to b_1, a_1, a_4 yields:

$$b_1 = \frac{12 - b_3^2}{4 b_3}; \tag{125}$$

$$a_1 = \frac{3}{2} - \frac{1}{4} b_3; \tag{126}$$

$$a_4 = \frac{6}{b_3}. \tag{127}$$

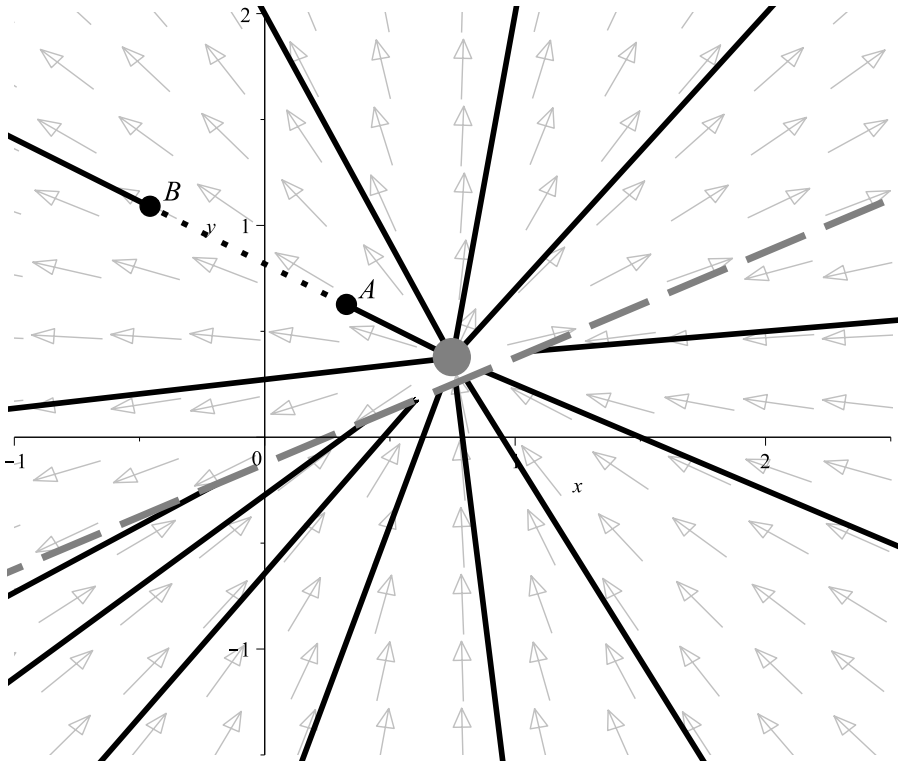


FIGURE 3. Phase plot of (112). Black lines denote kink solution trajectories. The gray circle denotes the unstable node (110). The gray dashed line denotes the equilibrium line (111). Gray arrows denote the direction field. The dotted line illustrates that perturbations in infected cell population y lead to proportional changes in uninfected cell population x. As the solution evolves from point A to B, y increases by 0.46, while x decreases by 1.09.

Condition (125) can be illustrated by the following numerical experiment. Let $\widetilde{x}\left(\tau; c, u, v; a_1, a_4, b_1, b_3\right)$, $\widetilde{y}\left(\tau; c, u, v; a_1, a_4, b_1, b_3\right)$ denote the numerical solution of (121) obtained by a forward constant-step integrator for arbitrary values of a_1, a_4, b_1, b_3. Error Δ between the numerical solution and kink solution $x\left(\tau; c, u, v; a_1, a_4, b_1, b_3\right)$, $y\left(\tau; c, u, v; a_1, a_4, b_1, b_3\right)$ in the form (4), (5) with parameters defined in Corollary 3 reads:

$$\Delta\left(a_1, a_4, b_1, b_3\right) = \sum_{j=0}^{N} \Big(\left|\widetilde{x}\left(c + jh; c, u, v; a_1, a_4, b_1, b_3\right) - x\left(c + jh; c, u, v; a_1, a_4, b_1, b_3\right)\right| + \left|\widetilde{y}\left(c + jh; c, u, v; a_1, a_4, ; b_1, b_3\right) - y\left(c + jh; c, u, v; a_1, a_4, b_1, b_3\right)\right| \Big), \tag{128}$$

where h is the integrator step size; c, u, v are any fixed initial conditions. Plot of error (128) when (126) and (127) hold true is given in Fig. 4. Note that the error is almost zero on the hyperbola defined by (125), which verifies results of Theorem (4.1). Analogous plots generated for conditions (126) and (127) are depicted in

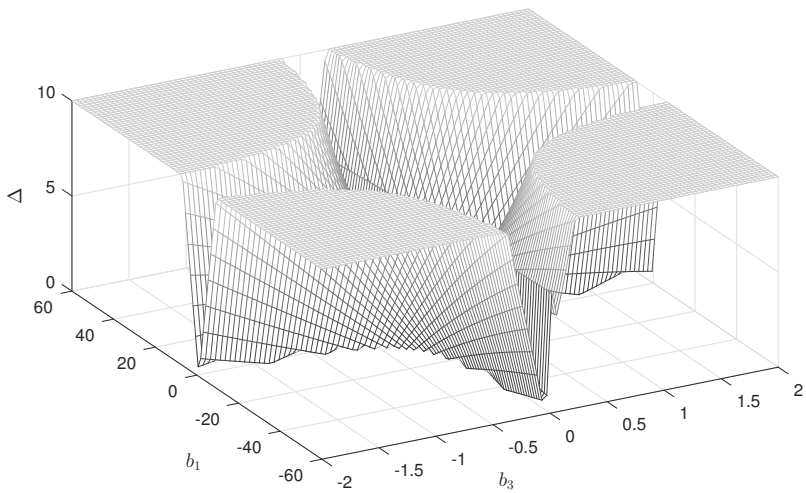


FIGURE 4. Plot of error (128) for $c = 0, u = 5, v = 1$. Conditions (126) and (127) hold true. The step size h is 10^{-4}; error is estimated over $N = 100$ steps. Errors higher than 10 are truncated to 10 for clarity. Note that the error is almost zero on the curve defined by (125).

Fig. 5 and 6. The classical Runge-Kutta 4th order method [10] is used to generate numerical solutions to (121).

5.2. **Kink solutions in hyperbolic relationship.**

5.2.1. *Equilibria.* Let Theorem 4.2 hold true. Equilibria of (3) read:

$$x_*^{(1)} = -\frac{a_4}{a_3}, \quad y_*^{(1)} = \frac{a_1 + b_1}{b_3}; \tag{129}$$

$$x_*^{(2)} = \alpha, \quad y_*^{(2)} = -\frac{a_1}{a_3}, \quad \alpha \in \mathbb{R}. \tag{130}$$

As in Subsection 5.1.1, (130) denotes a straight line of equilibrium points.

5.2.2. *Construction of kink solutions.* Consider the following system:

$$\begin{aligned} x'_\tau &= 3 + 2x + 2xy + 3y; \\ y'_\tau &= 3 + y - 2y^2; \\ x\Big|_{\tau=c} &= u, \quad y\Big|_{\tau=c} = v. \end{aligned} \tag{131}$$

Using transformation (6) on (112) results in:

$$\begin{aligned} \eta t \widehat{x}'_t &= 3 + 2\widehat{x} + 2\widehat{x}\widehat{y} + 3\widehat{y}; \\ \eta t \widehat{y}'_t &= 3 + \widehat{y} - 2\widehat{y}^2; \\ \widehat{x}\Big|_{t=\widehat{c}} &= u, \quad \widehat{y}\Big|_{t=\widehat{c}} = v. \end{aligned} \tag{132}$$

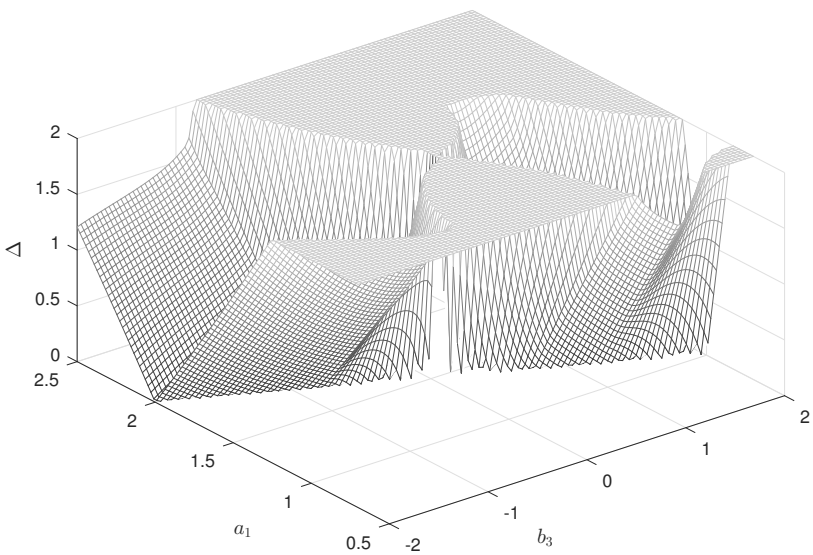


FIGURE 5. Plot of error (128) for $c = 0, u = 5, v = 1$. Conditions (125) and (127) hold true. The step size h is 10^{-3}; error is estimated over $N = 30$ steps. Errors higher than 2 are truncated to 2 for clarity. Note that the error is almost zero on the line defined by (126).

Theorem 4.2 yields the parameters of kink solutions to (132):

$$\lambda_1 = -\frac{1}{5}\left(2uv - 3u + 3v + 3\right); \quad \lambda_2 = \frac{1}{5}\left(v + 1\right)\left(2u + 3\right); \tag{133}$$

$$\mu_1 = -1; \quad \mu_2 = 1 + v; \quad \rho_x = -\frac{1}{\widehat{c}}; \quad \rho_y = \frac{2v - 3}{5\widehat{c}}. \tag{134}$$

Kink solutions to (132) are depicted in Fig. 7.

Corollary 5 yields that the above system has kink solutions (3) with the following parameters:

$$\eta = -5; \quad \sigma = -\frac{1}{5}\left(2uv - 3u + 3v + 3\right); \quad \gamma = -1; \quad \tau_x = 0; \tag{135}$$

$$\tau_y = \frac{2v + 2}{2v - 3}; \quad x_1 = \frac{2uv + 2u + 3v + 3}{2uv - 3u + 3v + 3}; \quad y_1 = -\frac{3v + 3}{2v - 3}. \tag{136}$$

Kink solutions to (131) are depicted in Fig. 8.

The phase plane of (131) can be seen in Fig. 9. Note that by Corollary 5, all solutions to (131) correspond to kink solutions in hyperbolic relationship:

$$xy - \frac{3}{2}x + \frac{3}{2}y = uv - \frac{3}{2}u + \frac{3}{2}v, \tag{137}$$

thus all phase trajectories are hyperbolas. Point (129) is a saddle point and the line (130) is repulsive in both half-planes.

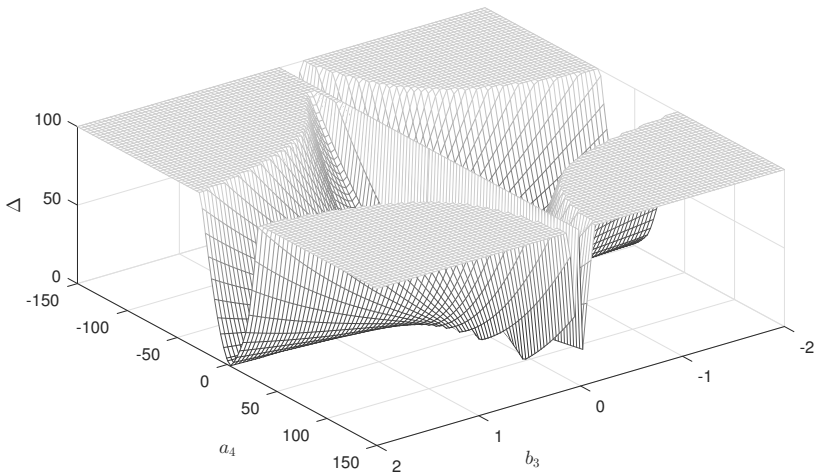
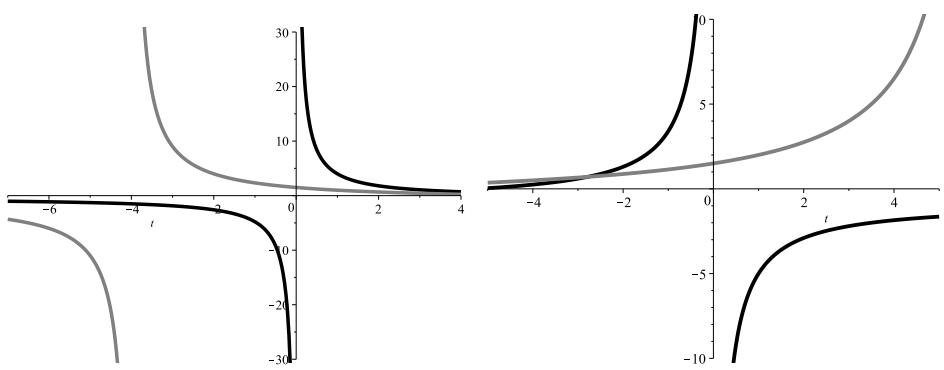


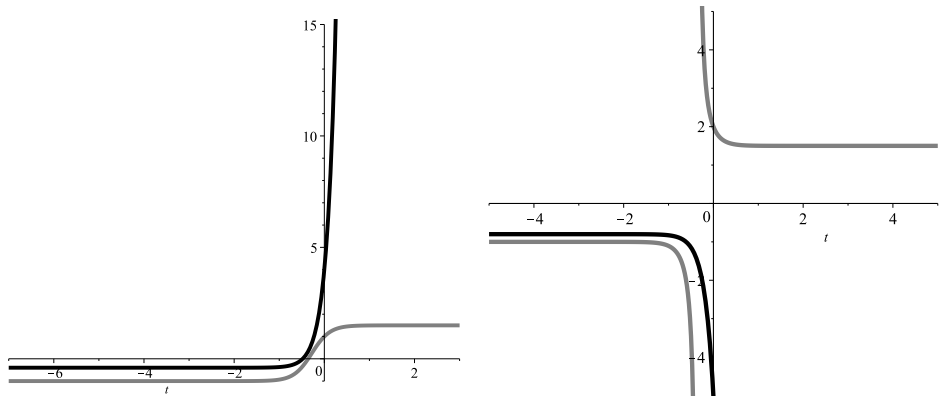
FIGURE 6. Plot of error (128) for $c = 0, u = 5, v = 1$. Conditions (125), (126) hold true. The step size h is 10^{-3}; error is estimated over $N = 30$ steps. Errors higher than 100 are truncated to 100 for clarity. Note that the error is almost zero on the hyperbola defined by (127).



(a) (b)

FIGURE 7. Kink solutions to (132) with $\widehat{c} = 1$. The black line denotes $\widehat{x}(t)$; the gray line denotes $\widehat{y}(t)$. In (a), $u = 4, v = 1$; in (b), $u = -5, v = 2$.

Note that if one kink solution is perturbed by an infinitesimal amount, the other solution can exhibit non-infinitesimal changes. For example, in Fig. 9, the variable representing infected cells y has been decreased by 5.19 from point A to B, which results in an increase of 0.39 in the variable representing uninfected cells x. Such instability under perturbations is observed for all systems (3) that satisfy the conditions of Theorem 4.2.



(a) (b)

FIGURE 8. Kink solutions to (131) with $c = 0$. The black line denotes $x(\tau)$; the gray line denotes $y(\tau)$. In (a), $u = 4, v = 1$; in (b), $u = -5, v = 2$.

6. **Concluding remarks.** Kink solitary solutions to a generalized system of hepatitis C evolution equations (3) coupled with both diffusive and multiplicative terms have been constructed in this paper. The generalized differential operator enabled the derivation of explicit existence conditions for kink solutions in terms of the system's parameters and the construction of general solutions to (3).

It has been shown that kink solutions to (3) hold for all initial conditions and can be in either linear or hyperbolic relationship. If kink solutions are in linear relationship, an infinitesimal perturbation of infected cell population results in an infinitesimal perturbation of uninfected cell population. However, if solutions are in hyperbolic relationship the former statement does not hold true – a large perturbation of infected cell population can lead to a infinitesimal alteration of uninfected cell population or vice versa. Such perturbation effects provide valuable insight into hepatitis C and other population evolution models.

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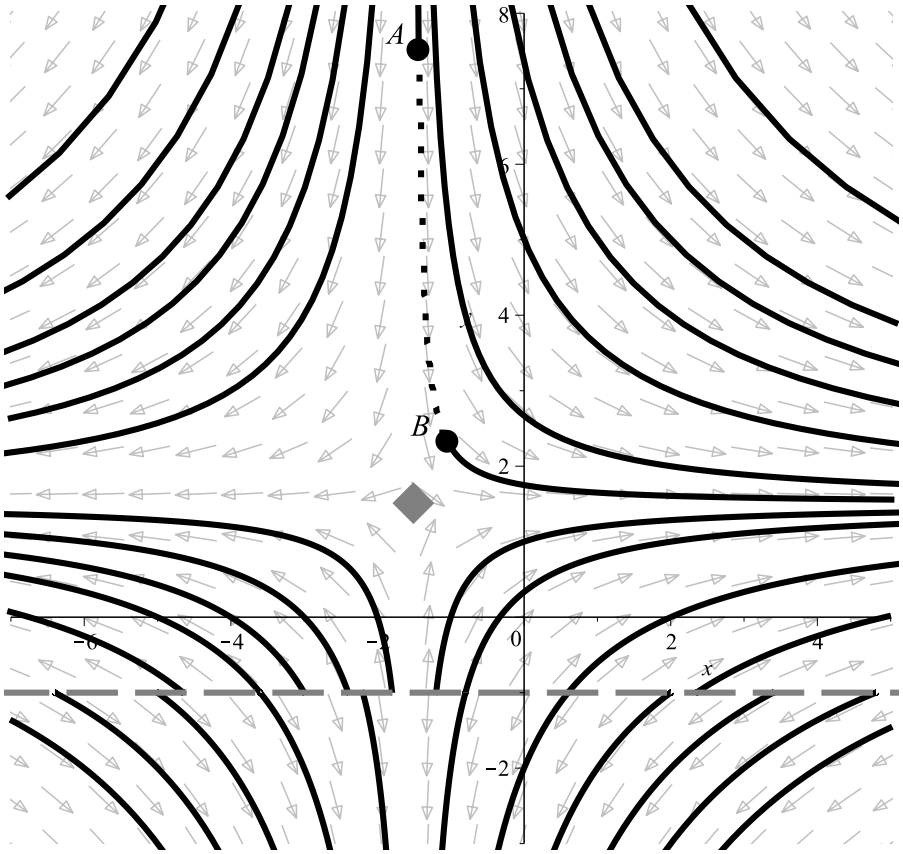

FIGURE 9. Phase plot of (131). Black lines denote kink solution trajectories. The gray diamond denotes the saddle point (129). The gray dashed line denotes the equilibrium line (130). Gray arrows denote direction field. The dotted line illustrates that large perturbations in infected cell population y lead to small changes in uninfected cell population x. As the solution evolves from point A to B, y decreases by 5.19, while x increases by 0.39.

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