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Existence of second order solitary solutions to Riccati differential equations coupled with a multiplicative term

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The construction of the second order solitary solutions to two Riccati differential equations coupled with a multiplicative term is discussed in this paper. The generalized differential operator method is used to derive necessary and sufficient conditions for the existence of these solutions in the space of system's parameters and initial conditions. The method does not need linearization, weak nonlinearity assumptions or perturbation theory. Computational experiments illustrate the complexity of dynamical processes in nonlinear differential equations coupled with a multiplicative term.

Keywords: Riccati equation; multiplicative coupling; solitary solution.

1. Introduction

Ordinary differential equations coupled with multiplicative terms is a common mathematical model describing the dynamics of nonlinear biological systems. The multiplicative coupling term is used to describe the air-borne disease transmission rate (the mass action term) in the model of a microparasite in a host population with the Allee effect (McCallum *et al.*, 2001; Regoes *et al.*, 2002; Courchamp *et al.*, 2008). Two nonlinear partial differential equations coupled with multiplicative terms are used to describe the spatio-temporal dynamics of a predator–prey system where the prey per capita growth rate is subject to the Allee effect in Petrovskii *et al.* (2005). The assumptions that per capita mortality rate of prey and predator are equal and that the rate of biomass production must be consistent with the rate of biomass assimilation help to simplify the model to a diffusive predator–prey system coupled with multiplicative terms in Owen & Lewis (2001). The travelling wave reduction transforms this model into a system of ordinary differential equations coupled with multiplicative terms (Kraenkel *et al.*, 2013):

$$\begin{aligned} y''_{xx} + hy'_x &= a_0 + a_1 y + a_2 y^2 + a_3 yz + a_4 y^3; \\ z''_{xx} + hz'_x &= b_0 + b_1 z + b_2 z^2 + b_3 yz + b_4 z^3, \end{aligned} \tag{1.1}$$

where $h, a_k, b_k \in \mathbb{R}, k = 0, \ldots, 4$; $a_4, b_4 \neq 0$. Analytical solitary solutions to (1.1) are derived in Kraenkel *et al.* (2013) and take the following form:

$$y(x) = \sigma \frac{\mathrm{e}^{\eta(x-c)} - y_1}{\mathrm{e}^{\eta(x-c)} - x_1}; \quad z(x) = \gamma \frac{\mathrm{e}^{\eta(x-c)} - z_1}{\mathrm{e}^{\eta(x-c)} - x_1}, \tag{1.2}$$

where $\eta, \sigma, \gamma, c, x_1, y_1, z_1 \in \mathbb{R}$; $\eta, \sigma, \gamma \neq 0$. Kraenkel *et al.* (2013) note that (1.2) is not the general solution to (1.1). These solitary solutions exist only at a preselected set of parameter values of (1.1); the (G'/G) expansion method is used to identify these parameters in Kraenkel *et al.* (2013). Simple computational experiments show that the variety of solutions to (1.1) is much more complex than (1.2). Note that $y(x)$ in (1.2) (also $z(x)$ in (1.2)) represents the general solution to the Riccati equation with constant coefficients (Polyanin & Zaitsev, 2003) if only these two Riccati equations are uncoupled.

But the coupled system of Riccati equations with multiplicative terms:

$$\begin{aligned} y'_x &= a_0 + a_1 y + a_2 y^2 + a_3 yz; \\ z'_x &= b_0 + b_1 z + b_2 z^2 + b_3 yz, \end{aligned} \tag{1.3}$$

has a much richer variety of solutions than (1.2). The main objective of this paper is to seek necessary and sufficient conditions for the existence of more complex second order solitary solutions to (1.3):

$$y(x) = \sigma \frac{\left(\mathrm{e}^{\eta(x-c)} - y_1\right)\left(\mathrm{e}^{\eta(x-c)} - y_2\right)}{\left(\mathrm{e}^{\eta(x-c)} - x_1\right)\left(\mathrm{e}^{\eta(x-c)} - x_2\right)}; \quad z(x) = \gamma \frac{\left(\mathrm{e}^{\eta(x-c)} - z_1\right)\left(\mathrm{e}^{\eta(x-c)} - z_2\right)}{\left(\mathrm{e}^{\eta(x-c)} - x_1\right)\left(\mathrm{e}^{\eta(x-c)} - x_2\right)}, \tag{1.4}$$

where $x_1, x_2, y_1, y_2, z_1, z_2$ are in general complex parameters. Such solutions (if only they exist) would enrich the mathematical description of solitary processes in nonlinear systems coupled with multiplicative terms and provide a deeper insight into the dynamics of biological systems.

But, instead of a straightforward parameter identification by using the (G'/G) expansion method (Kraenkel *et al.*, 2013), we do apply the generalized differential operator method (Navickas & Bikulciene, 2006; Navickas *et al.*, 2013) for the construction of a solution to (1.3).

A possible technique to determine the solution to (1.3) of the form (1.4) is to substitute the ansatz (1.4) into (1.3) and determine parameters so that the system is satisfied. This procedure, after gathering terms next to powers of $\mathrm{e}^{\eta(x-c)}$, yields, for the first equation of (1.4):

$$\begin{aligned}
&\Big(\gamma\sigma y_1 y_2 z_1 a_3 + \gamma\sigma y_1 y_2 z_2 a_3 + \gamma\sigma y_1 z_1 z_2 a_3 + \gamma\sigma y_2 z_1 z_2 a_3 + 2\sigma^2 y_1^2 y_2 a_2 + \\
&2\sigma^2 y_1 y_2^2 a_2 - \eta\sigma x_1 x_2 y_1 - \eta\sigma x_1 x_2 y_2 + \eta\sigma x_1 y_1 y_2 + \eta\sigma x_2 y_1 y_2 + \sigma x_1 x_2 y_1 a_1 + \\
&\sigma x_1 x_2 y_2 a_1 + \sigma x_1 y_1 y_2 a_1 + \sigma x_2 y_1 y_2 a_1 + 2 x_1^2 x_2 a_0 + 2 x_1 x_2^2 a_0\Big)\,\mathrm{e}^{-\eta(-x+c)} + \\
&\Big(-\gamma\sigma y_1 y_2 a_3 - \gamma\sigma y_1 z_1 a_3 - \gamma\sigma y_1 z_2 a_3 - \gamma\sigma y_2 z_1 a_3 - \gamma\sigma y_2 z_2 a_3 - \gamma\sigma z_1 z_2 a_3 - \\
&\sigma^2 y_1^2 a_2 - 4\sigma^2 y_1 y_2 a_2 - \sigma^2 y_2^2 a_2 + 2\eta\sigma x_1 x_2 - 2\eta\sigma y_1 y_2 - \sigma x_1 x_2 a_1 - \sigma x_1 y_1 a_1 - \\
&\sigma x_1 y_2 a_1 - \sigma x_2 y_1 a_1 - \sigma x_2 y_2 a_1 - \sigma y_1 y_2 a_1 - x_1^2 a_0 - 4 x_1 x_2 a_0 - x_2^2 a_0\Big)\,\mathrm{e}^{-2\eta(-x+c)} +
\end{aligned}$$

$$\Big(\gamma\,\sigma\,y_1\,a_3+\gamma\,\sigma\,y_2\,a_3+\gamma\,\sigma\,z_1\,a_3+\gamma\,\sigma\,z_2\,a_3+2\,\sigma^2y_1\,a_2+2\,\sigma^2y_2\,a_2-\eta\,\sigma\,x_1-\eta\,\sigma\,x_2+$$
$$\eta\,\sigma\,y_1+\eta\,\sigma\,y_2+\sigma\,x_1\,a_1+\sigma\,x_2\,a_1+\sigma\,y_1\,a_1+\sigma\,y_2\,a_1+2\,x_1\,a_0+2\,x_2\,a_0\Big)\,\mathrm{e}^{-3\,\eta\,(-x+c)}+$$
$$\Big(-\gamma\,\sigma\,a_3-\sigma^2a_2-\sigma\,a_1-a_0\Big)\,\mathrm{e}^{-4\,\eta\,(-x+c)}-\gamma\,\sigma\,y_1\,y_2\,z_1\,z_2\,a_3-\sigma^2{y_1}^2{y_2}^2a_2-$$
$$\sigma\,x_1\,x_2\,y_1\,y_2\,a_1-{x_1}^2{x_2}^2a_0=0. \tag{1.5}$$

Equation (1.5) is a system of nonlinear algebraic equations in respect of $\eta, x_1, x_2, \sigma, y_1, y_2, \gamma, z_1, z_2$ (the second equation of (1.3) yields similar results, which are omitted for brevity). It is clear that the solution of (1.5) with respect to the parameters of (1.4) becomes too complex to provide meaningful results.

A large variety of analytical techniques have been used for the derivation of exact solitary-wave solutions to nonlinear differential equations. Darboux transformation is employed to derive exact soliton solutions for the core of dispersion managed solitons in Xu *et al.* (2003). Eigenstate solitary-wave solutions to nonlinear Schrödinger equations are derived in Morgan *et al.* (1997); bright and dark algebraic solitary-wave solutions are derived in Hayata & Koshiba (1995). The series solution method is obtained to derive surface acoustic solitons of the Sakuma–Nishiguchi equation in Huang (1991).

In this paper we present the inverse balancing procedure for the derivation of the necessary conditions for the existence of solutions to (1.3) of the form (1.4). The generalized differential operator method is used to construct the second order solitary solutions to the system of coupled Riccati equations. These solutions (and the conditions of their existence in the space of system parameters and initial conditions) provide an insight into the complexity of transient processes in nonlinear systems coupled with multiplicative terms.

The main result of this paper is presented as Theorem 4.1. In this theorem, the necessary and sufficient conditions for the existence of solutions (1.4) in the system (1.3) in terms of the system parameters $a_k, b_k, k=0,\dots,3$ and the solution parameters $\sigma, \gamma, x_l, y_l, z_l, l=1,2$ are derived.

2. Preliminaries

Several definitions and statements which will be exploited in the process of the search of second order solitary solutions of (1.3) are concisely presented in this section.

2.1. *The operator expression of the general solution to a differential equation*

Suppose $P_k := P_k(c,s,t), k=1,2,3$ are analytic in c,s,t. Partial differentiation operators with respect to index variables are denoted as $\mathrm{D}_c, \mathrm{D}_s, \mathrm{D}_t$ (Navickas *et al.*, 2013).

Definition 2.1 The linear operator

$$\mathrm{D} := P_1\mathrm{D}_c + P_2\mathrm{D}_s + P_3\mathrm{D}_t, \tag{2.1}$$

is called the generalized differential operator.

Conventional differentiation properties hold for the generalized differential operator D (Navickas *et al.*, 2010a).

Suppose an ordinary differential equation is given:

$$y'_x = P(x, y), \tag{2.2}$$

where $P = P(c, s)$ is analytic in c and s. The general solution to (2.2) is $y = y(x, c, s)$, subject to the initial condition at $x = c$:

$$y(c, c, s) = s. \tag{2.3}$$

The generalized differential operator of (2.2) is defined as (Navickas & Bikulciene, 2006):

$$\mathrm{D} := \mathrm{D}_c + P(c, s)\mathrm{D}_s. \tag{2.4}$$

The general solution to (2.2) has the operator expression (Navickas & Bikulciene, 2006):

$$y = y(x, c, s) = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!}\mathrm{D}^j s. \tag{2.5}$$

2.2. *Linear recurring sequences and their canonical expressions*

The Hankel mapping of a sequence of functions $\left(w_j(c, s, t); j \in \mathbb{Z}_0\right)$ reads (Partington, 1988):

$$d_m = d_m(c, s, t) := \det \left(w_{j+k-2}\right)_{1 \le j,k \le m+1}, \quad m \in \mathbb{N}. \tag{2.6}$$

Note that there exists at least one $m_0 \in \mathbb{N}$ such that $d_{m_0} \neq 0$, if only there is at least one nonzero element in $\left(w_j(c, s, t); j \in \mathbb{Z}_0\right)$.

Suppose there exists such $m \in \mathbb{N}$ that $d_m \neq 0$, but $d_{m+k} = 0; k = 1, 2, \ldots$ for all values of $c, s, t \in \mathbb{R}$. In this case the sequence is a linear recurring sequence (Kurakin *et al.*, 1995) and has an H-rank equal to m, denoted as:

$$\mathrm{Hr}\left(w_j(c, s, t); j \in \mathbb{Z}_0\right) := m. \tag{2.7}$$

Note that the condition (2.7) implies the existence of an explicit linear recurrence of the following form:

$$w_{j+m} = \beta_1 w_{j+m-1} + \beta_2 w_{j+m-2} + \cdots + \beta_m w_j, \quad \beta_k \in \mathbb{R};\ j = 0, 1, \ldots. \tag{2.8}$$

Equation (2.8) can be verified by noting that $d_{m+k} = 0; k = 1, 2, \ldots$ implies that one row (or column) of the Hankel determinant can be taken as a linear combination (with coefficients $\beta_1, \ldots, \beta_m$) of the remaining rows (columns).

However, the explicit linear recurrence relation (2.8) is not the only way to represent linear recurring sequences. If (2.7) holds, it is possible to reconstruct the mathematical model of the sequence in the canonical form (Navickas & Bikulciene, 2006). The algorithm comprises the following steps:

1. Determination of the roots $\rho_k = \rho_k(c, s, t); k = 0, \dots, M$ of the characteristic equation:

$$\begin{vmatrix} w_0 & w_1 & \dots & w_m \\ w_1 & w_2 & \dots & w_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m-1} & w_m & \dots & w_{2m-1} \\ 1 & \rho & \dots & \rho^m \end{vmatrix} = 0. \tag{2.9}$$

   The multiplicity of root ρ_k is denoted $l_k; k = 1, \dots, M$ with $l_0 + l_1 + \dots + l_M = m$.

2. Expression of the sequence in canonical form:

$$w_j = \sum_{k=1}^{M} \sum_{r=0}^{l_k - 1} \mu_{kr}(c, s, t) \binom{j}{r} \rho_k^{j-r}(c, s, t); \quad j \in \mathbb{Z}_0. \tag{2.10}$$

   The following conventions are used in (2.10):

$$\binom{j}{r} = \begin{cases} \frac{j!}{r!(j-r)!}, & j \geq r; \\ 0, & j < r; \end{cases} \qquad j, r \in \mathbb{Z}_0. \tag{2.11}$$

   Also, $0^0 := 1, 0^k = 0, k \in \mathbb{N}$. The product $\binom{j}{r}\rho_k^{j-r}(c, s, t) = 0$ if at least one of the multipliers is equal to zero.

   The coefficients $\mu_{kr} = \mu_{kr}(c, s, t)$ are computed from the system of linear equations:

$$\sum_{k=1}^{M} \sum_{r=0}^{l_k - 1} \binom{j}{r} \rho_k^{j-r} \mu_{kr} = w_j; j = 0, 1, \dots, m - 1. \tag{2.12}$$

   Note that the system (2.12) has one and only one solution, because $d_m \neq 0$.

Note that not only (2.7) implies the existence of the canonical form (2.10), but the converse is also true: if the sequence $(w_j; j \in \mathbb{Z}_0)$ can be written in the form (2.10), it is a linear recurring sequence and satisfies (2.7).

REMARK 2.1 Suppose a sequence $(w_1, w_2, \dots)$ is given. Let

$$w_j = \sum_{k=1}^{m} \mu_k \rho_k^j, \quad j = 1, 2, \dots, \quad \rho_k \neq 0. \tag{2.13}$$

By the properties of linear recurring sequences, $(w_j; j \in \mathbb{N})$ has an H-rank of m:

$$\text{Hr } (w_1, w_2, \dots) = m. \tag{2.14}$$

Suppose any function ξ is given. It is possible to construct a linear recurring sequence $(\widetilde{w}_0, \widetilde{w}_1, \widetilde{w}_2, \dots)$ in such a way that:

$$\widetilde{w}_0 = \xi, \quad \widetilde{w}_j = w_j, j = 1, 2, \dots \tag{2.15}$$

To do this, denote $\rho_0 := 0$ and

$$\mu_0 := \xi - \sum_{k=1}^{m} \mu_k. \tag{2.16}$$

Defining

$$\widetilde{w}_j := \sum_{k=0}^{m} \mu_k \rho_k^j, \quad j = 0, 1, \dots \tag{2.17}$$

yields a sequence that satisfies (2.15), because $0^0 = 1$. Furthermore,

$$\mathrm{Hr}\ (\widetilde{w}_0, \widetilde{w}_1, \widetilde{w}_2, \dots) = m + 1. \tag{2.18}$$

The property noted in Remark 2.1 can be used in order to simplify the computation of Hankel determinants (2.6). In other words, if $\rho_0 = 0$ then one may consider the sequence starting with the element w_1.

2.3. *The construction of linear recurring sequences using the generalized differential operator*

Let a generalized differential operator D be given. Suppose a function $\rho := \rho(c, s, t)$ satisfies the conditions

$$\mathrm{D}\rho = \rho^2, \quad \mathrm{D}\lambda = \lambda\rho, \tag{2.19}$$

with some function $\lambda := \lambda(c, s, t)$, which is referred to as the coefficient of ρ (Navickas *et al.*, 2010a). Note that since $\mathrm{D}0 = 0 = 0^2$, the function $\rho(c, s, t) \equiv 0$ satisfies the conditions (2.19) with any λ that satisfies $\mathrm{D}\lambda = 0$.

Suppose several functions $\rho_k = \rho_k(c, s, t)$ satisfying (2.19) are given with their coefficients $\lambda_k = \lambda_k(c, s, t); k = 1, 2, \dots, m$. A sequence $\left(w_j; j \in \mathbb{Z}_0\right)$ can be constructed:

$$w_0(c, s, t) := \sum_{k=1}^{m} \lambda_k(c, s, t); \quad w_j(c, s, t) := j! \sum_{k=1}^{m} \lambda_k(c, s, t) \rho_k^j(c, s, t); \quad j = 1, 2, \dots. \tag{2.20}$$

The following recurrence equality holds for all sequences with elements defined by (2.20):

$$\mathrm{D}w_j = w_{j+1}; \quad w_j = \mathrm{D}^j w_0,\ j = 0, 1, \dots, \tag{2.21}$$

because

$$\mathrm{D}w_j = j! \sum_{k=1}^{m} \mathrm{D}\left(\lambda_k \rho_k^j\right) = j! \sum_{k=1}^{m} \left(\rho_k \lambda_k \rho_k^j + \lambda_k j \rho_k^{j-1} \rho_k^2\right) = (j+1)! \sum_{k=1}^{m} \lambda_k \rho_k^{j+1} = w_{j+1}, \tag{2.22}$$

where $\mathrm{D}^0 := \mathrm{I}$ is the identity operator.

The sequence defined by (2.20) is not a linear recurring sequence, because the multiplier $j!$ prohibits it from satisfying the canonical form (2.10). The multiplier $j!$ is necessary to insure that conditions (2.21) hold to make the connection with the series solution (2.5) given by the generalized differential operator. However, it can be noted that $\left(\frac{1}{j!}w_j; j \in \mathbb{Z}_0\right)$ is a linear recurring sequence, because

$$\frac{1}{j!} w_j = \sum_{k=1}^{m} \lambda_k \rho_k^j, \tag{2.23}$$

thus it satisfies (2.10) and

$$\mathrm{Hr}\left(\frac{1}{j!} w_j; j \in \mathbb{Z}_0\right) = m. \tag{2.24}$$

The functions $\rho_k = \rho_k(c, s, t)$ which satisfy conditions (2.19) are referred to as the common ratios of the generalized differential operator D.

3. Inverse balancing of the system parameters

A number of methods for the construction of exact solutions of nonlinear differential equations based on the extensive use of symbolic computations have been developed during the last decades. Homogeneous balance method (Wang, 1996), the Exp-function method (He & Wu, 2006; Ebaid, 2007), the tanh method (Malfeit, 1992; Li *et al.*, 2003) and its various extensions (Fan, 2000; Li *et al.*, 2003; Wazwaz, 2007), the (G'/G) expansion method (Zhang, 2010), the auxiliary ordinary differential equation method (Yomba, 2000; Sirendaoreji, 2000; Zhang, 2009) are successfully used to solve nonlinear differential equations with the help of computer algebra. The key idea of most of these methods is that the form of the solution of a nonlinear differential equation can be guessed (supposed) as a polynomial (or a ratio of polynomials) of standard functions whose argument is a traveling wave term. The degree of the polynomial can be determined by considering homogenous balance between the highest derivatives and nonlinear terms in the considered nonlinear differential equation. However, it can be noted that a straightforward application of these methods has attracted a considerable amount of criticism (Kudryashov & Loguinova, 2009; Navickas *et al.*, 2010b; Kudryashov, 2009a,b; Navickas & Ragulskis, 2009; Popovych, 2010).

In this section we consider the inverse balancing of the system parameters: from known parameters of the solitary solution we compute the parameters of the system of nonlinear differential equations that has the considered solution.

3.1. *The simplification of (1.4)*

The following functions are introduced in order to simplify the expression of (1.4):

$$X(x) := (x - x_1)(x - x_2); \;\; Y(x) := (x - y_1)(x - y_2); \;\; Z(x) := (x - z_1)(x - z_2), \tag{3.1}$$

where $x_1, x_2, y_1, y_2, z_1, z_2$ are, in general, complex parameters. Note that

$$X(x_1) = X(x_2) = Y(y_1) = Y(y_2) = Z(z_1) = Z(z_2) = 0;$$
$$X(y_1)X(y_2) = Y(x_1)Y(x_2), \dots, Y(z_1)Y(z_2) = Z(y_1)Z(y_2). \tag{3.2}$$

Then (1.4) reads:

$$y_0(x) = \sigma \frac{Y\left(\mathrm{e}^{\eta(x-c)}\right)}{X\left(\mathrm{e}^{\eta(x-c)}\right)}; \quad z_0(x) = \gamma \frac{Z\left(\mathrm{e}^{\eta(x-c)}\right)}{X\left(\mathrm{e}^{\eta(x-c)}\right)}. \tag{3.3}$$

The goal is to find a set of parameters of (1.3) which is necessary for (1.4) to satisfy (1.3).

The independent variable substitution

$$\widehat{x} := \mathrm{e}^{\eta x}; \quad x = \frac{1}{\eta} \ln \widehat{x}, \tag{3.4}$$

where $\eta \in \mathbb{R}/\{0\}$ is introduced. Equation (3.4) yields:

$$y(x) = y\left(\frac{1}{\eta}\ln\widehat{x}\right) =: \widehat{y}\,(\widehat{x}) = \widehat{y}\,(\mathrm{e}^{\eta x}); \quad z(x) = z\left(\frac{1}{\eta}\ln\widehat{x}\right) =: \widehat{z}\,(\widehat{x}) = \widehat{z}\,(\mathrm{e}^{\eta x}); \tag{3.5}$$

and

$$y'_x = \eta \widehat{x}\widehat{y}'_{\widehat{x}}; \quad z'_x = \eta \widehat{x}\widehat{z}'_{\widehat{x}}, \tag{3.6}$$

thus (3.3) is expressed as:

$$\widehat{y}_0 = \widehat{y}_0\,(\widehat{x}) = \sigma \frac{Y\left(\frac{\widehat{x}}{\widehat{c}}\right)}{X\left(\frac{\widehat{x}}{\widehat{c}}\right)}; \quad \widehat{z}_0 = \widehat{z}_0\,(\widehat{x}) = \gamma \frac{Z\left(\frac{\widehat{x}}{\widehat{c}}\right)}{X\left(\frac{\widehat{x}}{\widehat{c}}\right)}, \tag{3.7}$$

while (1.3) reads:

$$\eta\widehat{x}\,(\widehat{y}_0)'_{\widehat{x}} = a_0 + a_1\widehat{y}_0 + a_2\widehat{y}_0^2 + a_3\widehat{y}_0\widehat{z}_0;$$
$$\eta\widehat{x}\,(\widehat{z}_0)'_{\widehat{x}} = b_0 + b_1\widehat{z}_0 + b_2\widehat{z}_0^2 + b_3\widehat{y}_0\widehat{z}_0. \tag{3.8}$$

We refer to (3.7) and (3.8) as the images of (3.3) and (1.3), respectively.

3.2. *Second order solitary solutions to a coupled system of Riccati equations*

LEMMA 3.1 The second order solitary solution pair (1.4) satisfies the system of Riccati equations coupled with multiplicative terms (1.3) only if the following conditions hold true:

$$a_3 = b_2; \quad b_3 = a_2, \tag{3.9}$$

and

$$\frac{X(y_1)}{X(y_2)} = -\frac{y_1}{y_2}; \quad \frac{X(z_1)}{X(z_2)} = -\frac{z_1}{z_2}. \tag{3.10}$$

The system of Riccati differential equations (1.3) is called consistent if (3.9) is satisfied. The solitary solution pair (1.4) is called consistent if its parameters x_1 and x_2 satisfy (3.10).

Proof. Observe that (3.3) satisfies (1.3) if and only if (3.7) satisfies (3.8). Suppose that (3.7) satisfies (3.8). After inserting the expression of (3.7) to (3.8) and simplifying, two pairs of polynomials are obtained:

$$\begin{aligned}
&\eta\sigma\frac{\widehat{x}}{\widehat{c}}\left(\left(2\frac{\widehat{x}}{\widehat{c}}-(y_1+y_2)\right)X\left(\frac{\widehat{x}}{\widehat{c}}\right)-\left(2\frac{\widehat{x}}{\widehat{c}}-(x_1+x_2)\right)Y\left(\frac{\widehat{x}}{\widehat{c}}\right)\right)\\
&=a_0X^2\left(\frac{\widehat{x}}{\widehat{c}}\right)+a_1\sigma Y\left(\frac{\widehat{x}}{\widehat{c}}\right)X\left(\frac{\widehat{x}}{\widehat{c}}\right)+a_2\sigma^2Y^2\left(\frac{\widehat{x}}{\widehat{c}}\right)+a_3\sigma\gamma Y\left(\frac{\widehat{x}}{\widehat{c}}\right)Z\left(\frac{\widehat{x}}{\widehat{c}}\right);\\
&\eta\gamma\frac{\widehat{x}}{\widehat{c}}\left(\left(2\frac{\widehat{x}}{\widehat{c}}-(z_1+z_2)\right)X\left(\frac{\widehat{x}}{\widehat{c}}\right)-\left(2\frac{\widehat{x}}{\widehat{c}}-(x_1+x_2)\right)Z\left(\frac{\widehat{x}}{\widehat{c}}\right)\right)\\
&=b_0X^2\left(\frac{\widehat{x}}{\widehat{c}}\right)+b_1\gamma Z\left(\frac{\widehat{x}}{\widehat{c}}\right)X\left(\frac{\widehat{x}}{\widehat{c}}\right)+b_2\gamma^2Z^2\left(\frac{\widehat{x}}{\widehat{c}}\right)+b_3\gamma\sigma Z\left(\frac{\widehat{x}}{\widehat{c}}\right)Y\left(\frac{\widehat{x}}{\widehat{c}}\right).
\end{aligned} \tag{3.11}$$

Taking sequentially $\widehat{x}=\widehat{c}x_1;\widehat{x}=\widehat{c}x_2;\widehat{x}=\widehat{c}y_1;\widehat{x}=\widehat{c}y_2;\widehat{x}=\widehat{c}z_1;\widehat{x}=\widehat{c}z_2$ and substituting (3.2) into (3.11) we obtain linear system of equations in respect of parameters $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3,$
x_1, x_2:

$$\begin{array}{cl}
\text{(i)} & \sigma Y(x_1)a_2+\gamma Z(x_1)a_3=\eta x_1(x_2-x_1);\\
\text{((ii)} & \sigma Y(x_2)a_2+\gamma Z(x_2)a_3=\eta x_2(x_1-x_2);\\
\text{(iii)} & \gamma Z(x_1)b_2+\sigma Y(x_1)b_3=\eta x_1(x_2-x_1);\\
\text{(iv)} & \gamma Z(x_2)b_2+\sigma Y(x_2)b_3=\eta x_2(x_1-x_2);\\
\text{(v)} & a_0X^2(y_1)=\eta\sigma y_1(y_1-y_2)X(y_1);\\
\text{(vi)} & a_0X^2(y_2)=\eta\sigma y_2(y_2-y_1)X(y_2);\\
\text{(vii)} & b_0X^2(z_1)=\eta\gamma z_1(z_1-z_2)X(z_1);\\
\text{(viii)} & b_0X^2(z_2)=\eta\gamma z_2(z_2-z_1)X(z_2);
\end{array} \tag{3.12}$$

The first four equations (i)–(iv) of (3.12) yields the solution:

$$a_3=b_2=\Xi_2\frac{\eta}{\gamma}; \quad b_3=a_2=\Theta_2\frac{\eta}{\sigma}, \tag{3.13}$$

where

$$\Xi_2=\frac{(x_1-x_2)(x_2Y(x_1)+x_1Y(x_2))}{Y(x_1)Z(x_2)-Y(x_2)Z(x_1)}; \quad \Theta_2=\frac{(x_2-x_1)(x_1Z(x_2)+x_2Z(x_1))}{Y(x_1)Z(x_2)-Y(x_2)Z(x_1)}. \tag{3.14}$$

Note that degenerate situations when the denominators of algebraic expressions in (3.14) are equal to zero are not considered in this paper.

Equations (v)–(viii) of (3.12) yield (3.10). □

REMARK 3.1 Equation (3.10) can be rewritten as:

$$\begin{aligned} &2y_1y_2\left(x_1+x_2\right)-x_1x_2\left(y_1+y_2\right)=y_1y_2\left(y_1+y_2\right);\\ &2z_1z_2\left(x_1+x_2\right)-x_1x_2\left(z_1+z_2\right)=z_1z_2\left(z_1+z_2\right). \end{aligned} \tag{3.15}$$

Now, using the notations:

$$x_1+x_2=-\frac{\varDelta_1}{\varDelta_0};\quad x_1x_2=\frac{\varDelta_2}{\varDelta_0}, \tag{3.16}$$

where

$$\begin{aligned} \varDelta_0 &:= 2\left(y_1y_2\left(z_1+z_2\right)-z_1z_2\left(y_1+y_2\right)\right);\\ \varDelta_1 &:= \left(y_1+y_2\right)\left(z_1+z_2\right)\left(y_1y_2-z_1z_2\right);\\ \varDelta_2 &:= 2y_1y_2z_1z_2\left(\left(y_1+y_2\right)-\left(z_1+z_2\right)\right), \end{aligned} \tag{3.17}$$

yield:

$$x_{1,2}=\frac{1}{2\varDelta_0}\left(\varDelta_1\pm\sqrt{\varDelta_1^2-4\varDelta_0\varDelta_2}\right)\in\mathbb{C}; \tag{3.18}$$

$$a_0=\varTheta_0\eta\sigma;\quad b_0=\varXi_0\eta\gamma, \tag{3.19}$$

where

$$\varTheta_0=\frac{y_1\left(y_1-y_2\right)}{X\left(y_1\right)};\quad \varXi_0=\frac{z_1\left(z_1-z_2\right)}{X\left(z_1\right)}. \tag{3.20}$$

Taking $\widehat{x}=0$ in (3.11) yields:

$$\begin{aligned} &\left(x_1x_2\right)^2a_0+\sigma x_1x_2y_1y_2a_1+\sigma^2\left(y_1y_2\right)^2a_2+\sigma\gamma y_1y_2z_1z_2a_3=0;\\ &\left(x_1x_2\right)^2b_0+\gamma x_1x_2z_1z_2b_1+\gamma^2\left(z_1z_2\right)^2b_2+\gamma\sigma y_1y_2z_1z_2b_3=0. \end{aligned} \tag{3.21}$$

Equalities (3.13), (3.18), (3.19) and (3.21) yield:

$$a_1=-\left(\varTheta_0+\varTheta_2+\varXi_2\right)\eta;\quad b_1=-\left(\varXi_0+\varXi_2+\varTheta_2\right)\eta. \tag{3.22}$$

COROLLARY 3.1 If (3.3) satisfies (1.3) then the system must have the following form:

$$\begin{aligned} \left(y_0\right)_x' &= \varTheta_0\eta\sigma-\left(\varTheta_0+\varTheta_2+\varXi_2\right)\eta y_0+\varTheta_2\frac{\eta}{\sigma}y_0^2+\varXi_2\frac{\eta}{\gamma}y_0z_0;\\ \left(z_0\right)_x' &= \varXi_0\eta\gamma-\left(\varXi_0+\varXi_2+\varTheta_2\right)\eta z_0+\varXi_2\frac{\eta}{\gamma}z_0^2+\varTheta_2\frac{\eta}{\sigma}y_0z_0. \end{aligned} \tag{3.23}$$

Let us consider that the parameters $(\eta, \sigma, y_1, y_2, \gamma, z_1, z_2)$ of (3.3) are fixed. Then auxiliary parameters $(x_1, x_2, \Theta_0, \Theta_2, \Xi_0, \Xi_2)$ can be computed using the described algorithm, which also yields the parameters $(a_0, a_1, a_2, b_0, b_1, b_2)$ of (1.3).

In the remainder of this paper, only Riccati systems (1.3) and solitary solutions (1.4) that do satisfy the conditions of Lemma 3.1 will be considered.

The system that satisfies the conditions of Lemma 3.1 reads:

$$\begin{aligned} y'_x &= a_0 + a_1 y + a_2 y^2 + b_2 yz; \\ z'_x &= b_0 + b_1 z + b_2 z^2 + a_2 yz. \end{aligned} \tag{3.24}$$

It is possible to reduce the number of parameters in (3.24) by rescaling. Dividing both equations of (3.24) by b_2 yields:

$$\begin{aligned} \frac{1}{b_2} y'_x &= \widetilde{a}_0 + \widetilde{a}_1 y + \widetilde{a}_2 y^2 + yz; \\ \frac{1}{b_2} z'_x &= \widetilde{b}_0 + \widetilde{b}_1 z + z^2 + \widetilde{a}_2 yz, \end{aligned} \tag{3.25}$$

where $\widetilde{a}_k = \frac{a_k}{b_2}, k = 0, 1, 2$ and $\widetilde{b}_k = \frac{b_k}{b_2}, k = 0, 1$. Introducing a new independent variable $\widetilde{x} = \frac{x}{b_2}$ yields:

$$\begin{aligned} y(x) &= y\left(b_2 \widetilde{x}\right) = \widetilde{y}\left(\widetilde{x}\right); \quad \widetilde{y}'_{\widetilde{x}} = \frac{1}{b_2} y'_x, \\ z(x) &= z\left(b_2 \widetilde{x}\right) = \widetilde{z}\left(\widetilde{x}\right); \quad \widetilde{z}'_{\widetilde{x}} = \frac{1}{b_2} z'_x. \end{aligned} \tag{3.26}$$

Using (3.26) in (3.25) yields:

$$\begin{aligned} \widetilde{y}'_{\widetilde{x}} &= \widetilde{a}_0 + \widetilde{a}_1 \widetilde{y} + \widetilde{a}_2 \widetilde{y}^2 + \widetilde{y}\widetilde{z}; \\ \widetilde{z}'_{\widetilde{x}} &= \widetilde{b}_0 + \widetilde{b}_1 \widetilde{z} + \widetilde{z}^2 + \widetilde{a}_2 \widetilde{y}\widetilde{z}. \end{aligned} \tag{3.27}$$

The number of parameters in the system has now been reduced from 6 to 5. However, similar rearrangements cannot be used to reduce the number of parameters further. Also, given a solution $\widetilde{y}, \widetilde{z}$, the construction of the system (1.3) would be problematic, because the scaling parameter b_2 would be unknown. For these reasons in the remaining parts of the paper we consider the system (1.3) which satisfies (3.9) directly.

3.3. *Relations between the parameters of solitary solutions and the parameters of the Riccati system*

Formally, the inverse balancing of system parameters can described by the following steps:

1. Computation of auxiliary parameters:

$$(y_1, y_2, z_1, z_2) \to (x_1, x_2, \Theta_0, \Theta_2, \Xi_0, \Xi_2). \tag{3.28}$$

2. Computation of the parameters of the system of differential equations:

$$(\eta, \sigma, y_1, y_2, \gamma, z_1, z_2) \to (a_0, a_1, a_2, b_0, b_1, b_2). \tag{3.29}$$

Equations (3.14), (3.10) and (3.19) yield:

$$(\psi y_1, \psi y_2, \psi z_1, \psi z_2) \to (\psi x_1, \psi x_2, \Theta_0, \Theta_2, \Xi_0, \Xi_2)\,;\ \ \psi \in \mathbb{R}\backslash\{0\}. \tag{3.30}$$

Equations (3.14), (3.10) and (3.23) yield:

$$(\phi\eta, \alpha\sigma, \psi y_1, \psi y_2, \beta\gamma, \psi z_1, \psi z_2) \to \left(\phi\alpha a_0, \phi a_1, \frac{\phi}{\alpha} a_2, \phi\beta b_0, \phi b_1, \frac{\phi}{\beta} b_2\right), \tag{3.31}$$

with $\phi, \alpha, \beta, \psi \in \mathbb{R}\backslash\{0\}$.

Equations (3.30) and (3.31) together with:

$$\begin{aligned} y_0(x) &= \sigma \frac{y_1 y_2}{x_1 x_2} \frac{\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{y_1}\right)\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{y_2}\right)}{\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{x_1}\right)\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{x_2}\right)}; \\ z_0(x) &= \gamma \frac{z_1 z_2}{x_1 x_2} \frac{\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{z_1}\right)\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{z_2}\right)}{\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{x_1}\right)\left(\mathrm{e}^{-\eta(x-c)} - \frac{1}{x_2}\right)}, \end{aligned} \tag{3.32}$$

yield:

$$\left(-\eta, \sigma\frac{y_1 y_2}{x_1 x_2}, \frac{1}{y_1}, \frac{1}{y_2}, \gamma\frac{z_1 z_2}{x_1 x_2}, \frac{1}{z_1}, \frac{1}{z_2}\right) \to (a_0, a_1, a_2, b_0, b_1, b_2)\,. \tag{3.33}$$

3.4. *Phase plane trajectories of second order solitary solutions*

LEMMA 3.2 Each trajectory of the second order solitary solution (1.4) in the phase plane (y, z) satisfies the general equation of a conic section:

$$\widehat{A}y^2 + 2\widehat{B}yz + \widehat{C}z^2 + \widehat{D}y + \widehat{E}z = 1. \tag{3.34}$$

Proof. Inserting (3.7) into (3.34) and multiplying by $X\left(\frac{\hat{x}}{\hat{c}}\right)$ yields the polynomial equality:

$$\begin{aligned} &\sigma^2 Y^2\left(\frac{\widehat{x}}{\widehat{c}}\right)\widehat{A} + 2\sigma\gamma Y\left(\frac{\widehat{x}}{\widehat{c}}\right) Z\left(\frac{\widehat{x}}{\widehat{c}}\right)\widehat{B} + \gamma^2 Z^2\left(\frac{\widehat{x}}{\widehat{c}}\right)\widehat{C} + \\ &\sigma Y\left(\frac{\widehat{x}}{\widehat{c}}\right) X\left(\frac{\widehat{x}}{\widehat{c}}\right)\widehat{D} + \gamma Z\left(\frac{\widehat{x}}{\widehat{c}}\right) X\left(\frac{\widehat{x}}{\widehat{c}}\right) - X^2\left(\frac{\widehat{x}}{\widehat{c}}\right) = 0. \end{aligned} \tag{3.35}$$

Taking sequentially $\widehat{x} = \widehat{c}x_1; \widehat{x} = \widehat{c}x_2; \widehat{x} = \widehat{c}y_1; \widehat{x} = \widehat{c}y_2; \widehat{x} = \widehat{c}z_1; \widehat{x} = \widehat{c}z_2$ and substituting into (3.35) the following linear system with respect to $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}, \widehat{E}$ is obtained:

$$\begin{array}{ll}
\text{(i)} & \gamma^2 Z^2(y_1)\widehat{C} + \gamma Z(y_1) X(y_1)\widehat{E} = X^2(y_1); \\
\text{(ii)} & \gamma^2 Z^2(y_2)\widehat{C} + \gamma Z(y_2) X(y_2)\widehat{E} = X^2(y_2); \\
\text{(iii)} & \sigma^2 Y^2(z_1)\widehat{A} + \sigma Y(z_1) X(z_1)\widehat{D} = X^2(z_1); \\
\text{(iv)} & \sigma^2 Y^2(z_2)\widehat{A} + \sigma Y(z_2) X(z_2)\widehat{D} = X^2(z_2); \\
\text{(v)} & \sigma^2 Y^2(x_1)\widehat{A} + 2\sigma\gamma Y(x_1) Z(x_1)\widehat{B} + \gamma^2 Z^2(x_1)\widehat{C} = 0; \\
\text{(vi)} & \sigma^2 Y^2(x_2)\widehat{A} + 2\sigma\gamma Y(x_2) Z(x_2)\widehat{B} + \gamma^2 Z^2(x_2)\widehat{C} = 0.
\end{array} \tag{3.36}$$

In the nondegenerate case (when the conic section is not transformed into a line), (3.36) has a rank equal to five, with solution:

$$\begin{aligned}
&\widehat{A} = -\frac{X(z_1) X(z_2)}{\sigma^2 Y(z_1) Y(z_2)}; \quad \widehat{C} = -\frac{X(y_1) X(y_2)}{\gamma^2 Z(y_1) Z(y_2)}; \\
&\widehat{B} = \frac{Y(x_1) Z(x_2) + Y(x_2) Z(x_1)}{\sigma\gamma\left(Y(z_1) Y(z_2) + Z(y_1) Z(y_2)\right)}; \\
&\widehat{D} = \frac{Y(z_1) X(z_2) + Y(z_2) X(z_1)}{\sigma Y(z_1) Y(z_2)}; \quad \widehat{E} = \frac{Z(y_1) X(y_2) + Z(y_2) X(y_1)}{\gamma Z(y_1) Z(y_2)}.
\end{aligned} \tag{3.37}$$

□

The discriminant of (3.34) is expressed as:

$$\widehat{A}\widehat{C} - \widehat{B}^2 = -\frac{\left(Z(x_2) Y(x_1) - Y(x_2) Z(x_1)\right)^2}{4\left(Y(z_1) Y(z_2)\right)^2}. \tag{3.38}$$

Thus, the type of the conic section (3.34) depends on the sign of

$$\tau = -\left(Z(x_2) Y(x_1) - Y(x_2) Z(x_1)\right)^2, \tag{3.39}$$

because the denominator of (3.38) is positive, but the numerator is the square of either a real or imaginary number in the general case. This claim can be verified by symbolic computations using computer algebra, which show that the sign of τ is equal to the sign of τ_0, defined as:

$$\begin{aligned}
\tau_0 := 4\Delta_0\Delta_2 - \Delta_1^2 = 16 y_1 y_2 z_1 z_2 \left((y_1 + y_2) - (z_1 + z_2)\right) \Big(y_1 y_2 (z_1 + z_2) \\
- z_1 z_2 (y_1 + y_2)\Big) + (y_1 + y_2)^2 (z_1 + z_2)^2 (y_1 y_2 - z_1 z_2)^2.
\end{aligned} \tag{3.40}$$

Equation (3.34) is a fragment of an ellipse if $\tau_0 > 0$; a fragment of a parabola if $\tau_0 = 0$; a fragment of a hyperbola if $\tau_0 < 0$.

4. The generalized differential operator method for the system of Riccati equations

Necessary conditions for the existence of solution (1.4) to the system (1.3) were derived in the previous section. If these conditions are satisfied, a second order solitary solution can be derived by substituting the ansatz (1.4) into (1.3) and determining the parameters $\eta, \sigma, \gamma, x_k, y_k, z_k$, $k = 1, 2$ such that the system is satisfied, however, this procedure results only in a particular solution to (1.3) of the form (1.4).

In this section a technique for determining the general solution to (1.3) is proposed. Sufficient conditions for the existence of a general solution to (1.3) of the form (1.4) are derived using the generalized differential operator method (Navickas & Bikulciene, 2006; Navickas *et al.*, 2010a).

4.1. *Special expressions of the determinant of the Hankel matrix*

The image of a consistent system of Riccati equations coupled with multiplicative terms is given by:

$$\begin{aligned} \eta\widehat{x}\,(\widehat{y})'_{\widehat{x}} &= a_0 + a_1\widehat{y} + a_2\widehat{y}^2 + b_2\widehat{y}\widehat{z}; \\ \eta\widehat{x}\,(\widehat{z})'_{\widehat{x}} &= b_0 + b_1\widehat{z} + b_2\widehat{z}^2 + a_2\widehat{y}\widehat{z}. \end{aligned} \tag{4.1}$$

The general solution to (4.1), reads:

$$\widehat{y} = \widehat{y}\,(\widehat{x}, \widehat{c}, s, t)\,; \quad \widehat{z} = \widehat{z}\,(\widehat{x}, \widehat{c}, s, t), \tag{4.2}$$

and is subject to initial conditions

$$\widehat{y}\,(\widehat{c}, \widehat{c}, s, t) = s; \quad \widehat{z}\,(\widehat{c}, \widehat{c}, s, t) = t, \tag{4.3}$$

and is expressible in the operator form:

$$\widehat{y} = \sum_{j=0}^{+\infty} \frac{(\widehat{x} - \widehat{c})^j}{j!} \mathrm{D}^j s; \quad \widehat{z} = \sum_{j=0}^{+\infty} \frac{(\widehat{x} - \widehat{c})^j}{j!} \mathrm{D}^j t, \tag{4.4}$$

where D is the generalized differential operator of (4.1) defined as (Navickas *et al.*, 2013):

$$\mathrm{D} := \mathrm{D}_{\widehat{c}} + \frac{1}{\eta\widehat{c}}\left(\left(a_0 + a_1 s + a_2 s^2 + b_2 st\right)\mathrm{D}_s + \left(b_0 + b_1 t + b_2 t^2 + a_2 st\right)\mathrm{D}_t\right). \tag{4.5}$$

Let

$$\widehat{p}_j := \frac{1}{j!}\mathrm{D}^j s; \quad \widehat{q}_j := \frac{1}{j!}\mathrm{D}^j t; \quad j = 0, 1, \ldots. \tag{4.6}$$

Hankel matrix determinants are computed:

$$W_{\widehat{y}} := W_{\widehat{y}}\,(\widehat{c}, s, t, \eta) = \begin{vmatrix} \widehat{p}_1 & \widehat{p}_2 & \widehat{p}_3 \\ \widehat{p}_2 & \widehat{p}_3 & \widehat{p}_4 \\ \widehat{p}_3 & \widehat{p}_4 & \widehat{p}_5 \end{vmatrix}; \quad W_{\widehat{z}} := W_{\widehat{z}}\,(\widehat{c}, s, t, \eta) = \begin{vmatrix} \widehat{q}_1 & \widehat{q}_2 & \widehat{q}_3 \\ \widehat{q}_2 & \widehat{q}_3 & \widehat{q}_4 \\ \widehat{q}_3 & \widehat{q}_4 & \widehat{q}_5 \end{vmatrix}. \tag{4.7}$$

Following Remark 2.1, the determinants in (4.7) are computed starting from $\widehat{p}_1$, not $\widehat{p}_0$. Note that while it is possible to compute the Hankel determinants starting from $\widehat{p}_0$, the expression of the determinants (and the subsequent characteristic polynomial) become complicated to deal with. The same procedure is used for the sequence $\left(\widehat{q}_j; j \in \mathbb{Z}_0\right)$.

If the solutions to (4.1) are second order solitary solution images (for all initial conditions $s, t \in \mathbb{R}$), it is necessary that determinants (4.7) are equal to zero (according to Section 2.2). This condition verifies that the sequences $\left(\widehat{p}_j; j \in \mathbb{Z}_0\right)$ and $\left(\widehat{q}_j; j \in \mathbb{Z}_0\right)$ are linear recurring sequences of the third order (the equality of (4.7) together with Remark 2.1 yield this statement). That it is necessary can be seen by considering the expression $\widehat{y}\left(\widehat{x}\right)$. Note that:

$$\widehat{y}\left(\widehat{x}\right) = \frac{Y\left(\frac{\widehat{x}}{\widehat{c}}\right)}{X\left(\frac{\widehat{x}}{\widehat{c}}\right)} = \frac{\left(\widehat{x} - \widehat{c}y_1\right)\left(\widehat{x} - \widehat{c}y_2\right)}{\left(\widehat{x} - \widehat{c}x_1\right)\left(\widehat{x} - \widehat{c}x_2\right)} = A + \frac{B}{1 - \left(\widehat{x} - \left(\widehat{c}\widehat{x_1} - 1\right)\right)} + \frac{C}{1 - \left(\widehat{x} - \left(\widehat{c}\,\widehat{x_2} - 1\right)\right)}. \tag{4.8}$$

The parameters A, B, C are unknown and can depend on the initial conditions s, t. Each term of the equation in (4.8) can be expanded as a series in powers of $\left(\widehat{x} - \widehat{c}\right)$ and each term corresponds to a term of the canonical expression of the linear recurring sequence $\left(\widehat{p}_j; j \in \mathbb{Z}_0\right)$, as shown in subsequent sections. Thus, there must exist $\eta_0 \in \mathbb{R}$ such that:

$$W_{\widehat{y}}\left(\widehat{c}, s, t, \pm\eta_0\right) = 0; \quad W_{\widehat{z}}\left(\widehat{c}, s, t, \pm\eta_0\right) = 0. \tag{4.9}$$

Condition (4.9) holds true if:

$$9a_0a_1a_2 + 9b_0b_1b_2 - 18a_0a_2b_1 - 18b_0b_2a_1 + 3a_1b_1^2 + 3b_1a_1^2 - 2a_1^3 - 2b_1^3 = 0. \tag{4.10}$$

Furthermore,

$$\eta_0^2 = \frac{a_1^2 - a_1b_1 + b_1^2}{3} - a_0a_2 - b_0b_2. \tag{4.11}$$

Detailed derivations of (4.10) and (4.11) are given in Appendix A.

4.2. *The construction of the canonical expression of linear recurring sequences and verification of sufficient conditions*

Without loss of generality we assume that $\eta = \eta_0$ in this and all subsequent subsections.

The roots of the characteristic equations $\widehat{\rho}_1, \widehat{\rho}_2$ of sequences $\left(\widehat{p}_j; j \in \mathbb{Z}_0\right)$ and $\left(\widehat{q}_j; j \in \mathbb{Z}_0\right)$ are computed by solving algebraic equations:

$$\begin{vmatrix} \widehat{p}_1 & \widehat{p}_2 & \widehat{p}_3 \\ \widehat{p}_2 & \widehat{p}_3 & \widehat{p}_4 \\ 1 & \widehat{\rho} & \widehat{\rho}^2 \end{vmatrix} = 0; \quad \begin{vmatrix} \widehat{q}_1 & \widehat{q}_2 & \widehat{q}_3 \\ \widehat{q}_2 & \widehat{q}_3 & \widehat{q}_4 \\ 1 & \widehat{\rho} & \widehat{\rho}^2 \end{vmatrix} = 0. \tag{4.12}$$

Computer algebra helps to prove that both equalities in (4.12) have the same solutions $\widehat{\rho}_k := \widehat{\rho}_k(c, s, t)$, $k = 1, 2$ of the form

$$\widehat{\rho}_k\left(\widehat{c}, s, t\right) = \frac{1}{\widehat{c}}\rho_k(s, t); \quad k = 1, 2, \tag{4.13}$$

where $\rho_k := \rho_k(s, t)$ depends only on s and t.

Equation (2.12) yields the coefficients:

$$\lambda_1 = \frac{\widehat{p}_2 - \widehat{p}_1\widehat{\rho}_2}{\widehat{\rho}_1\left(\widehat{\rho}_1 - \widehat{\rho}_2\right)}; \quad \lambda_2 = \frac{\widehat{p}_2 - \widehat{p}_1\widehat{\rho}_1}{\widehat{\rho}_2\left(\widehat{\rho}_2 - \widehat{\rho}_1\right)}, \tag{4.14}$$

for sequence $\left(\widehat{p}_j; j \in \mathbb{N}\right)$ and

$$\mu_1 = \frac{\widehat{q}_2 - \widehat{q}_1\widehat{\rho}_2}{\widehat{\rho}_1\left(\widehat{\rho}_1 - \widehat{\rho}_2\right)}; \quad \mu_2 = \frac{\widehat{q}_2 - \widehat{q}_1\widehat{\rho}_1}{\widehat{\rho}_2\left(\widehat{\rho}_2 - \widehat{\rho}_1\right)}, \tag{4.15}$$

for sequence $\left(\widehat{q}_j; j \in \mathbb{N}\right)$. Note that $\lambda_1, \lambda_2, \mu_1, \mu_2$ do not depend on $\widehat{c}$.

Thus, all parameters of the canonical expression for $\widehat{p}_j, \widehat{q}_j$ are computed and

$$\widehat{p}_j = \lambda_0 0^j + \lambda_1\rho_1^j + \lambda_2\rho_2^j; \quad \widehat{q}_j = \mu_0 0^j + \mu_1\rho_1^j + \mu_2\rho_2^j. \tag{4.16}$$

The following conditions are sufficient to show that (4.1) has solutions which are the images of second order solitary solutions for all values of $s, t \in \mathbb{R}$ (Navickas *et al.*, 2013):

$$\begin{aligned} &\text{(i)} \quad \lambda_1 + \lambda_2 = s - \sigma; \quad \mu_1 + \mu_2 = t - \gamma. \\ &\text{(ii)} \quad \mathrm{D}\widehat{\rho}_k = \widehat{\rho}_k^2; \quad \mathrm{D}\lambda_k = \lambda_k\widehat{\rho}_k; \quad \mathrm{D}\mu_k = \mu_k\widehat{\rho}_k, \end{aligned} \tag{4.17}$$

when $k = 1, 2$.

4.3. *The closed form of the second order solitary solution*

Inserting (4.16) into (4.4) yields the closed form of the general solution $\widehat{y} = \widehat{y}\left(\widehat{x}, \widehat{c}, s, t\right), \widehat{z} = \widehat{z}\left(\widehat{x}, \widehat{c}, s, t\right)$ to system (4.1):

$$\widehat{y} = \sigma + \frac{\lambda_1}{1 - \widehat{\rho}_1\left(\widehat{x} - \widehat{c}\right)} + \frac{\lambda_2}{1 - \widehat{\rho}_2\left(\widehat{x} - \widehat{c}\right)}; \quad \widehat{z} = \gamma + \frac{\mu_1}{1 - \widehat{\rho}_1\left(\widehat{x} - \widehat{c}\right)} + \frac{\mu_2}{1 - \widehat{\rho}_2\left(\widehat{x} - \widehat{c}\right)}. \tag{4.18}$$

By substitution (3.4), the solution of the respective consistent system

$$\begin{aligned} y'_x &= a_0 + a_1 y + a_2 y^2 + b_2 yz; \\ z'_x &= b_0 + b_1 z + b_2 z^2 + a_2 yz, \end{aligned} \tag{4.19}$$

has the form:

$$\begin{aligned} y = y(x, c, s, t) &= \sigma + \frac{\lambda_1(s,t)}{1 - \rho_1(s,t)\left(\mathrm{e}^{\eta_0(x-c)} - 1\right)} + \frac{\lambda_2(s,t)}{1 - \rho_2(s,t)\left(\mathrm{e}^{\eta_0(x-c)} - 1\right)}; \\ z = z(x, c, s, t) &= \gamma + \frac{\mu_1(s,t)}{1 - \rho_1(s,t)\left(\mathrm{e}^{\eta_0(x-c)} - 1\right)} + \frac{\mu_2(s,t)}{1 - \rho_2(s,t)\left(\mathrm{e}^{\eta_0(x-c)} - 1\right)}. \end{aligned} \tag{4.20}$$

Let $\sigma, \gamma \neq 0$. Then (4.20) can be transformed to the form:

$$y(x, c, s, t) = \sigma\frac{Y_*\left(\mathrm{e}^{\eta_0(x-c)}, s, t\right)}{X_*\left(\mathrm{e}^{\eta_0(x-c)}, s, t\right)}; \quad z(x, c, s, t) = \gamma\frac{Z_*\left(\mathrm{e}^{\eta_0(x-c)}, s, t\right)}{X_*\left(\mathrm{e}^{\eta_0(x-c)}, s, t\right)}, \tag{4.21}$$

where X_*, Y_*, Z_* are auxiliary functions defined as:

$$\begin{aligned} X_*(x,s,t) &:= (x - x_1(s,t))(x - x_2(s,t)); \\ Y_*(x,s,t) &:= (x - y_1(s,t))(x - y_2(s,t)); \\ Z_*(x,s,t) &:= (x - z_1(s,t))(x - z_2(s,t)). \end{aligned} \tag{4.22}$$

The functions $x_k(s,t), y_k(s,t), z_k(s,t); k = 1,2$ read:

$$x_k = 1 + \frac{1}{\rho_k}; y_{1,2} = \frac{1}{2}\left(L_y \pm \sqrt{L_y^2 - 4K_y}\right); \quad z_{1,2} = \frac{1}{2}\left(L_z \pm \sqrt{L_z^2 - 4K_z}\right), \tag{4.23}$$

where K_y, K_z, L_y, L_z are functions of s,t satisfying the equalities:

$$\begin{aligned} &y_1(s,t)y_2(s,t) = K_y(s,t); \quad z_1(s,t)z_2(s,t) = K_z(s,t); \\ &y_1(s,t) + y_2(s,t) = L_y(s,t); \quad z_1(s,t) + z_2(s,t) = L_z(s,t), \end{aligned} \tag{4.24}$$

and are explicitly given by:

$$\begin{aligned} K_y &:= K_y(s,t) = \left(1 + \frac{1}{\rho_1}\right)\left(1 + \frac{1}{\rho_2}\right) + \frac{\lambda_1}{\sigma\rho_1}\left(1 + \frac{1}{\rho_2}\right) + \frac{\lambda_2}{\sigma\rho_2}\left(1 + \frac{1}{\rho_1}\right); \\ K_z &:= K_z(s,t) = \left(1 + \frac{1}{\rho_1}\right)\left(1 + \frac{1}{\rho_2}\right) + \frac{\mu_1}{\gamma\rho_1}\left(1 + \frac{1}{\rho_2}\right) + \frac{\mu_2}{\gamma\rho_2}\left(1 + \frac{1}{\rho_1}\right); \\ L_y &:= L_y(s,t) = 2 + \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{\lambda_1}{\sigma\rho_1} + \frac{\lambda_2}{\sigma\rho_2}; \\ L_z &:= L_z(s,t) = 2 + \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{\mu_1}{\gamma\rho_1} + \frac{\mu_2}{\gamma\rho_2}. \end{aligned} \tag{4.25}$$

Note that the initial conditions s,t do satisfy the following relations:

$$\sigma\frac{Y_*(1,s,t)}{X_*(1,s,t)} = s; \quad \gamma\frac{Z_*(1,s,t)}{X_*(1,s,t)} = t. \tag{4.26}$$

The results of the preceding computations can be summarized with the following theorem:

THEOREM 4.1 The consistent Riccati system (4.19) has second order solitary solutions (1.4) iff (3.9), (3.10) and (4.10) hold.

The proof of the theorem follows from previous computations.

4.4. *Explicit expressions of invariants*

Computer algebra helps to identify the invariants of (4.1) (the kernel elements of (4.5)), denoted as $\mathrm{inv}_y, \mathrm{inv}_z, \mathrm{Inv}_y, \mathrm{Inv}_z \in \mathbb{R}$.

$$\begin{aligned}&\mathrm{Inv}_y := \frac{y_1(s,t)y_2(s,t)}{x_1(s,t)x_2(s,t)}, \quad \mathrm{Inv}_z := \frac{z_1(s,t)z_2(s,t)}{x_1(s,t)x_2(s,t)};\\ &\mathrm{inv}_y := \frac{\lambda_1(s,t)}{1+\rho_1(s,t)} + \frac{\lambda_2(s,t)}{1+\rho_2(s,t)}, \quad \mathrm{inv}_z := \frac{\mu_1(s,t)}{1+\rho_1(s,t)} + \frac{\mu_2(s,t)}{1+\rho_2(s,t)}.\end{aligned} \tag{4.27}$$

These invariants also satisfy the following equalities:

$$\mathrm{Inv}_y = 1 + \frac{1}{\sigma}\mathrm{inv}_y; \quad \mathrm{Inv}_z = 1 + \frac{1}{\gamma}\mathrm{inv}_z. \tag{4.28}$$

Invariants are employed in the computation of limits as $x \to \pm\infty$ and formal description of the generalized differential operator method.

It is worthwhile to observe that the operator method allows not only to compute the expression of the general solution, but also demonstrates the qualitative dependence of the solution on the initial conditions (Navickas *et al.*, 2010a,b).

Analogously to the method of balancing, the operator method can be formally described as:

$$(a_0, a_1, a_2, b_0, b_1, b_2) \to (\eta, \sigma, y_1(s,t), y_2(s,t), \gamma, z_1(s,t), z_2(s,t)). \tag{4.29}$$

For example, the operator method can describe the following mapping:

$$(a_0, a_1, a_2, b_0, b_1, b_2) \to \left(-\eta, \sigma\mathrm{Inv}_y, \frac{1}{y_1(s,t)}, \frac{1}{y_2(s,t)}, \gamma\mathrm{Inv}_z, \frac{1}{z_1(s,t)}, \frac{1}{z_2(s,t)}\right). \tag{4.30}$$

If (4.21) is the general solution to (4.19), then the function pair

$$y^+(x,c,s,t) = \sigma\frac{Y_*\left(-\mathrm{e}^{\eta_0(x-c)}, s, t\right)}{X_*\left(-\mathrm{e}^{\eta_0(x-c)}, s, t\right)}; \quad z^+(x,c,s,t) = \gamma\frac{Z_*\left(-\mathrm{e}^{\eta_0(x-c)}, s, t\right)}{X_*\left(-\mathrm{e}^{\eta_0(x-c)}, s, t\right)}, \tag{4.31}$$

is also a general solution to (4.19), that is with any $s_0, t_0 \in \mathbb{R}$ there exists such initial conditions $s_1, t_1 \in \mathbb{R}$ that $y_k(s_0,t_0) = -y_k(s_1,t_1)$; $z_k(s_0,t_0) = -z_k(s_1,t_1)$, $k = 1, 2$. It can be noted that, for concrete values of parameters s_0, t_0, the functions $y^+(x,c,s_0,t_0), z^+(x,c,s_0,t_0)$ give the other fragment of the conic section defined by the solution $y(x,c,s_0,t_0), z(x,c,s_0,t_0)$.

5. Computational experiments

5.1. *Equilibria*

The equilibrium points $\left(y_*^{(k)}, z_*^{(k)}\right)$ of (4.19) are determined by solving a system of algebraic equations

$$\begin{aligned}a_0 + a_1 y + a_2 y^2 + b_2 yz &= 0;\\ b_0 + b_1 y + b_2 y^2 + a_2 yz &= 0.\end{aligned} \tag{5.1}$$

The solution to (5.1) is given by:

$$y_*^{(k)} = -\frac{\left(z_*^{(k)}\right)^2 b_2 + z_*^{(k)} b_1 + b_0}{z_*^{(k)} a_2}; \quad z_*^{(k)} = z_*^{(k)}, \tag{5.2}$$

where $z_*^{(k)}$ is a root of the third order algebraic equation

$$(b_1 b_2 - a_1 a_2)\, z^3 + \left(b_1^2 + b_0 b_2 + a_0 a_2 - a_1 b_1\right) z^2 + (2b_0 b_1 - a_1 b_0)\, z + b_0^2 = 0. \tag{5.3}$$

The consistent system (4.19) has at most three equilibrium points. Suppose that (5.3) has three distinct real roots. The coordinates of two equilibrium points may be found using the general solution (4.21):

$$\begin{aligned} &\lim_{x \to +\infty} (y(x, c, s, t), z(x, c, s, t)) = (\sigma, \gamma) =: \left(y_*^{(1)}, z_*^{(1)}\right); \\ &\lim_{x \to -\infty} (y(x, c, s, t), z(x, c, s, t)) = \left(\sigma \mathrm{Inv}_y, \gamma \mathrm{Inv}_z\right) =: \left(y_*^{(2)}, z_*^{(2)}\right); \end{aligned} \tag{5.4}$$

for all values of c, s, t. Thus $\left(y_*^{(1)}, z_*^{(1)}\right) = (\sigma, \gamma)$ is a stable node and $\left(y_*^{(2)}, z_*^{(2)}\right) = \left(\sigma \mathrm{Inv}_y, \gamma \mathrm{Inv}_z\right)$ is an unstable node. The third equilibrium point $\left(y_*^{(3)}, z_*^{(3)}\right)$ is a saddle.

The coordinates of the saddle and its stable and unstable manifolds as well as the separatrix between elliptic and hyperbolic phase trajectories can be found analytically using computer algebra. This procedure is outlined in the examples of this section.

Note that the solutions to (4.19) with initial conditions that lie on the stable and unstable manifolds of the saddle point are not second order solitary solutions.

5.2. *Analysing the phase plane at infinity*

Since the solitary solutions (4.21) can have singularity points, the standard phase plane is not sufficient to fully explain the dynamics of the analysed system. To counter this, the procedure of mapping the phase plane to a diametrical plane of a hemisphere, that is described in detail in Jordan & Smith (2007) is used.

Figure 1 contains an illustration of the mapping. Suppose any point $P = (x_0, y_0)$ on the x, y plane is given. A hemisphere of radius $OO^* = R$ is drawn in such a way that is touches the x, y plane at O. A straight line PO^* is drawn that intersects the hemisphere at point P'. The projection of point P' onto the hemisphere's diametrical plane x^*, y^* yields the image of P that is denoted $P^* = (x_0^*, y_0^*)$.

Using elementary geometrical computations, it can be derived that:

$$x_0^* = \frac{x_0}{\sqrt{1 + \frac{r_0^2}{R^2}}}, \quad y_0^* = \frac{y_0}{\sqrt{1 + \frac{r_0^2}{R^2}}}; \quad r_0^2 = x_0^2 + y_0^2. \tag{5.5}$$

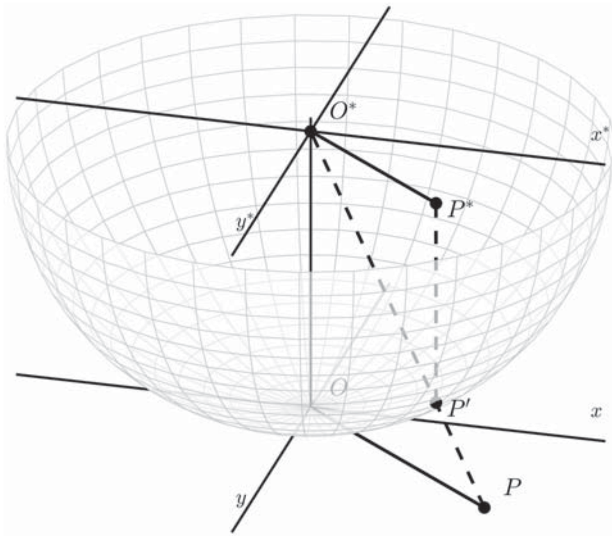

FIG. 1. Illustration of mapping the point P from the x, y plane to the point P^* on the diametrical plane x^*, y^* of the hemisphere.

5.3. *Computational examples*

EXAMPLE 5.1 Consider the system

$$\begin{aligned} y_x' &= \frac{136}{11} - \frac{828}{319}y + \frac{29}{187}y^2 - \frac{550}{1479}yz; \\ z_x' &= -\frac{51}{29} + \frac{345}{319}z - \frac{550}{1479}z^2 + \frac{29}{187}yz. \end{aligned} \tag{5.6}$$

The system (5.6) has equilibrium points at $(y_*^{(1)}, z_*^{(1)}) = (4, 3)$ (this equilibrium point is a stable node); $(y_*^{(2)}, z_*^{(2)}) = \left(\frac{34}{3}, \frac{17}{25}\right)$ (this equilibrium point is an unstable node) and $(y_*^{(3)}, z_*^{(3)}) = \left(\frac{136}{23}, \frac{51}{46}\right)$ (this equilibrium point is a saddle).

As can be seen from Figs 2 and 3, there are three distinct types of solutions that have either none, one or two singularities.

To derive the explicit expression of the separatrix between elliptic and hyperbolic solutions, the characteristic roots ρ_1, ρ_2 of system (5.6) are considered:

$$\begin{aligned} \rho_{1,2} = &-\frac{240}{319} + \frac{29s}{374} - \frac{275t}{1479} \\ &\pm \frac{\sqrt{6365529s^2 - 30528300st + 36602500t^2 - 171649782s - 312252600t + 1558425564}}{32538}. \end{aligned} \tag{5.7}$$

Note that $x_k = 1 + \frac{1}{\rho_k}, k = 1, 2$, thus x_k are complex numbers only when ρ_k are complex numbers. It can be observed that the phase trajectories are elliptic when both x_1 and x_2 are complex and hyperbolic

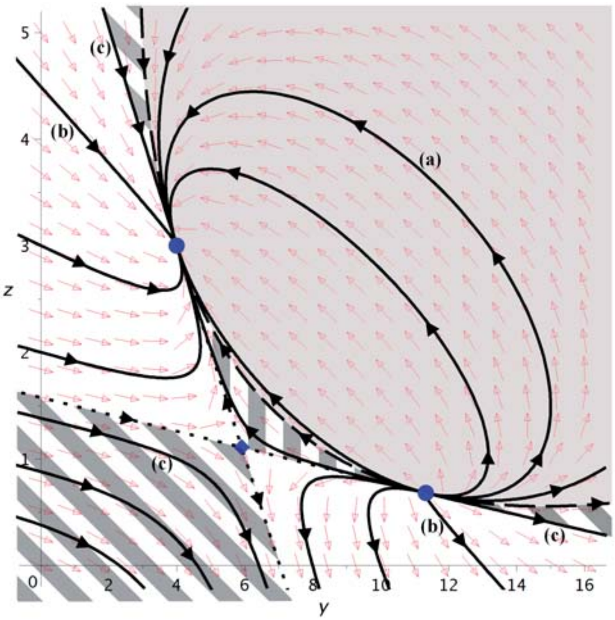

FIG. 2. The phase portrait of (5.6). Circles denote the nodes $(y_*^{(1)}, z_*^{(1)}) = (4, 3)$ and $(y_*^{(2)}, z_*^{(2)}) = \left(\frac{34}{3}, \frac{17}{25}\right)$; the diamond denotes the saddle $(y_*^{(3)}, z_*^{(3)}) = \left(\frac{136}{23}, \frac{51}{46}\right)$; black lines denote solution trajectories. Labels (a)–(c) correspond to respective solutions depicted in Fig. 3. The dashed line denotes the separatrix between elliptic and hyperbolic trajectories. The dotted line denotes the stable and unstable manifolds of the saddle point. The solution trajectories are ellipse fragments and have no singularities in the grey filled region. Solution trajectories are hyperbola fragments and have no singularities in the vertically striped region. Solution trajectories are hyperbola fragments with one singular point in the unfilled region. Solution trajectories are hyperbolas with two singular points in the diagonally striped region.

when both x_1 and x_2 are real. This yields the separatrix equation, produced by the equation in the square root:

$$6365529s^2 - 30528300st + 36602500t^2 - 171649782s - 312252600t + 1558425564 = 0. \quad (5.8)$$

It can be observed that degenerate cases of phase trajectories are produced when $x_k = \frac{\rho_k+1}{\rho_k}$ is equal to zero or undefined. Solving equations for both the numerator and denominator yields:

$$\begin{aligned} \rho_k + 1 = 0 &\Rightarrow 88t + 87s - 612 = 0; \\ \rho_k = 0 &\Rightarrow 1100t + 87s - 1734 = 0. \end{aligned} \quad (5.9)$$

It can be seen that the unstable and stable manifolds of the saddle point $\left(y_*^{(3)}, z_*^{(3)}\right)$ lie on the straight lines defined by (5.9), respectively, furthermore, the saddle point is located at the intersection of the two lines (Fig. 2).

Using the procedure outlined in Section 5.2, the solution trajectories that have singularity points can be analysed with respect to their divergence to infinity. The phase plane depicted in Fig. 2 is mapped to the diametrical plane of a hemisphere with radius $R = 10$ in Fig. 4. Note that in case (b) (the unfilled

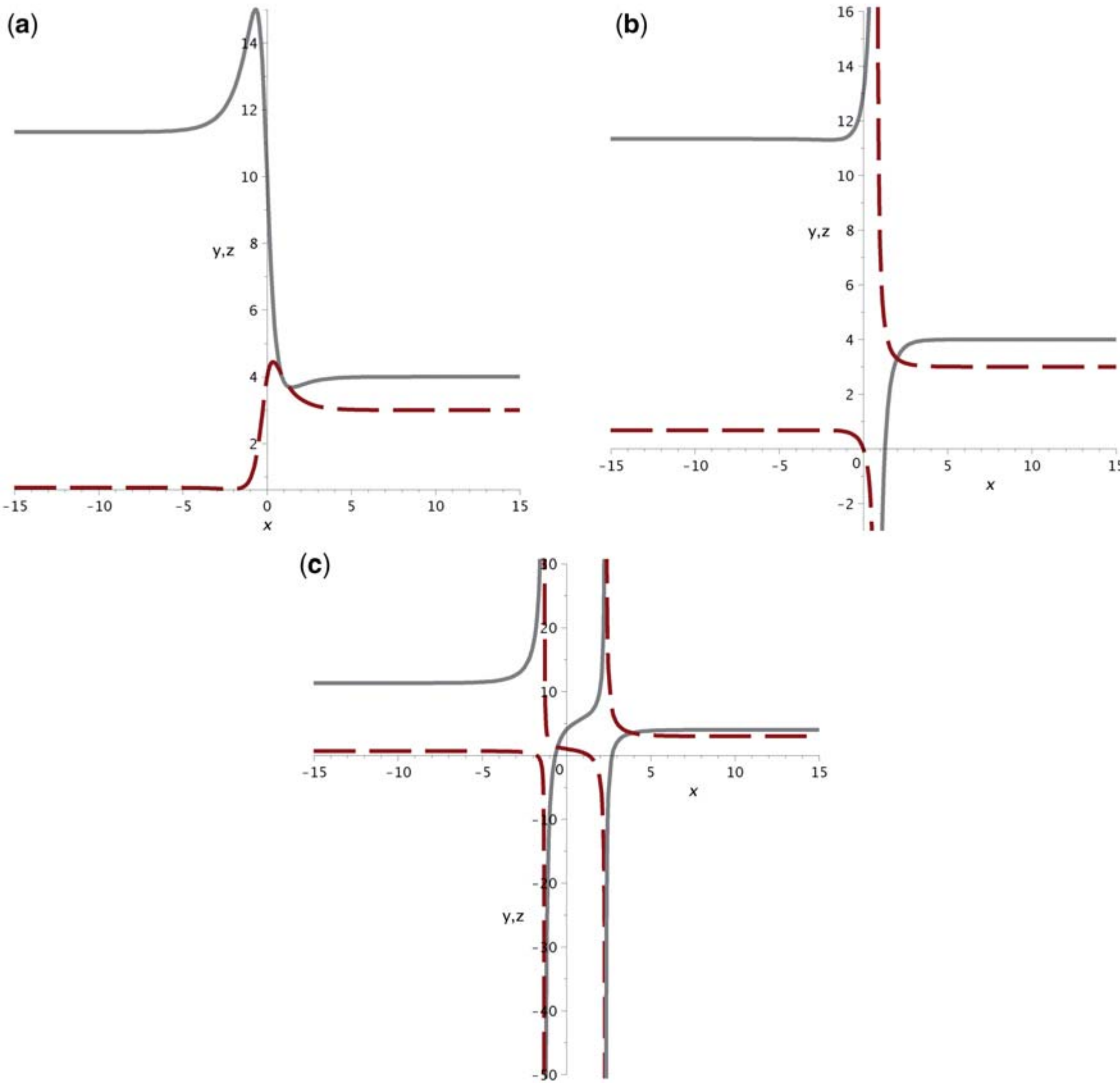

FIG. 3. The evolution of solutions to (5.6) with respect to x. The solid grey line denotes $y(x)$; the black dashed line denotes $z(x)$. Parts (a), (b) and (c) correspond to solutions to (5.6) with zero, one and two singularities respectively. Note that in the case (b), the phase plane trajectories meet only at infinity, while the same effect is observed twice in (c).

regions in Figs 2 and 4), the trajectory diverges to infinity away from the unstable node at point $B^{-\infty}$ in Fig. 4 and loops through to connect with the other part of the trajectory at point $B^{+\infty}$ that converges to the stable node. For case (c) (diagonally striped regions in Figs 2 and 4), the same effect is observed twice: the trajectory diverges from the unstable node at point $C_1^{-\infty}$, loops through infinity entering the '5 to 9 o'clock' region at point $C_1^{+\infty}$, and finally loops through infinity at point $C_2^{-\infty}$ to reappear at point $C_2^{+\infty}$ to converge to the stable node.

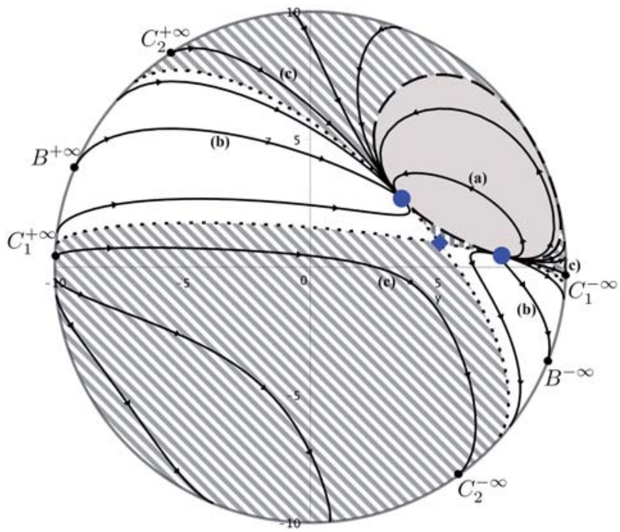

FIG. 4. The map of the phase plane depicted in Fig. 2 to the diametrical plane of a hemisphere with radius $R = 10$. All line, region and label representations remain the same as in Fig. 2. Labels (a)-(c) correspond to respective solutions depicted in Fig. 3. The transparent grey circle denotes infinite points with respect to the original phase plane. The points $B^{-\infty}, B^{+\infty}$ correspond to the singularity point of the solitary solution depicted in Fig. 3(b). Points $C_1^{-\infty}, C_1^{+\infty}, C_2^{-\infty}, C_2^{+\infty}$ correspond to the singularity points of the solitary solution depicted in Fig. 3(c).

Such analysis provides some insight into the nature of solitary solutions with singularities: it can be seen that they define heteroclinic orbits, since they do join two equilibrium points, however, they themselves are comprised of several segments that only 'connect' at infinity.

EXAMPLE 5.2 Consider the system

$$\begin{aligned} y'_x &= 6 - \frac{1}{12}y^2 + \frac{1}{10}yz; \\ z'_x &= -5 + \frac{1}{10}z^2 - \frac{1}{12}yz. \end{aligned} \tag{5.10}$$

The system (5.10) has two equilibrium points at $(y_*^{(1)}, z_*^{(1)}) = (6, -5)$ (this equilibrium point is a stable node) and $(y_*^{(2)}, z_*^{(2)}) = (-6, 5)$ (this equilibrium point is an unstable node).

The phase portrait of (5.10) is depicted in Fig. 5. It can be observed that the solutions to (5.10) can be classified into two distinct groups: solutions that remain bounded as $x \to \pm\infty$ (Fig. 6(a)) and solutions that have one singular point (Fig. 6(b)).

As in the previous example, the characteristic roots ρ_1, ρ_2 can be used to compute the separatrix between hyperbolic and elliptic phase trajectories. The characteristic roots read:

$$\rho_{1,2} = -\frac{1}{2} - \frac{s}{24} + \frac{t}{20} \pm \frac{\sqrt{(60 + 5s - 6t)\,(5s - 60 - 6t)}}{120}. \tag{5.11}$$

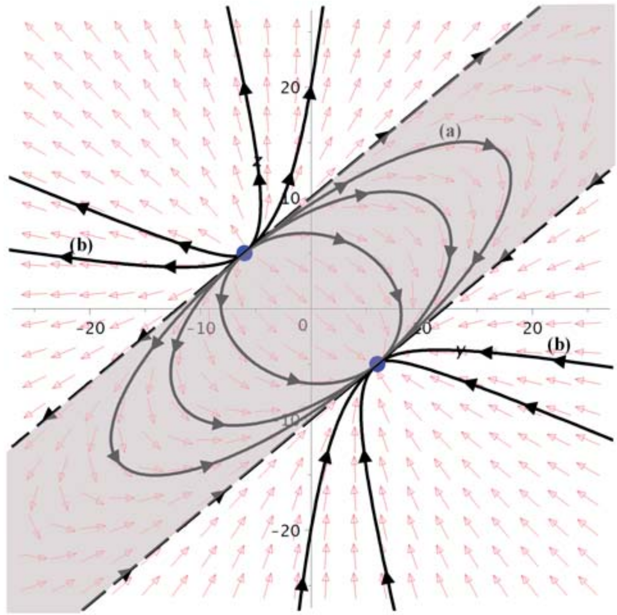

FIG. 5. The phase portrait of (5.10). Circles denote the nodes $(y_*^{(1)}, z_*^{(1)}) = (6, -5)$ and $(y_*^{(2)}, z_*^{(2)}) = (-6, 5)$; black lines denote solution trajectories. Labels (a) and (b) correspond to respective solutions depicted in Fig. 6. The dashed line corresponds to the separatrix between elliptic and hyperbolic trajectories. The solution trajectories are ellipse fragments and have no singularities in the grey-filled region; the solution trajectories are hyperbola fragments and have one singular point in the unfilled region.

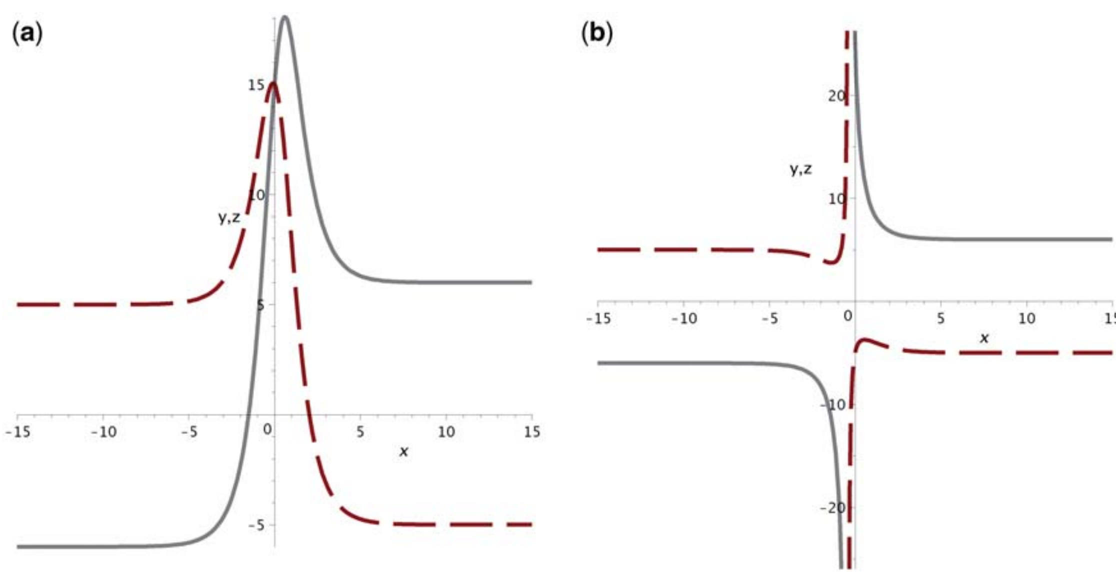

FIG. 6. The evolution of solutions to (5.10) with respect to x. The solid grey line denotes $y(x)$; the black dashed line denotes $z(x)$. Note that in the case (b), the trajectories meet in the phase plane only at infinity.

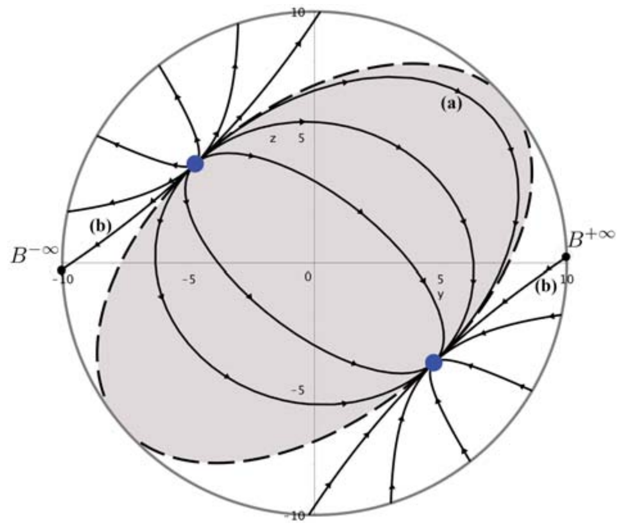

FIG. 7. The map of the phase plane depicted in Fig. 5 to the diametrical plane of a hemisphere with radius $R = 10$. All line, region and label representations remain the same as in Fig. 5. Labels (a) and (b) correspond to respective solutions depicted in Fig. 6. The transparent grey circle denotes infinite points with respect to the original phase plane. The points $B^{-\infty}, B^{+\infty}$ correspond to the singularity of the solitary solution depicted in Fig. 6(b).

In this case the separatrix is given by two straight lines:

$$5s - 6t + 60 = 0; \quad 5s - 6t - 60 = 0. \tag{5.12}$$

Note that there is no saddle point for this system, however, computing the degenerate cases yields

$$\begin{aligned} \rho_k + 1 = 0 &\Rightarrow 5s - 6t + 60 = 0; \\ \rho_k = 0 &\Rightarrow 5s - 6t - 60 = 0, \end{aligned} \tag{5.13}$$

which is congruent to (5.12), leading to the conclusion that the saddle point lies at infinity.

Projection of the phase plane depicted in Fig. 5 to the diametrical plane of a hemisphere was performed using the same algorithm as the previous example and is depicted in Fig. 7. The radius of the hemisphere was chosen to be $R = 10$ to provide maximum clarity.

For the system (5.10), solutions with two singularities do not exist. As in the previous example, solutions with one singularity define heteroclinic orbits between the unstable and stable nodes that diverges to infinity at point $B^{-\infty}$ and reappears at point $B^{+\infty}$ to converge to the stable node.

6. Concluding remarks

Necessary and sufficient conditions for the existence of the second order solitary solutions to a system of Riccati equations coupled with a multiplicative term are derived in this paper. The generalized differential operator method is used to construct the structure of the solutions and the conditions of their existence in the space of system parameters and initial conditions. The main result of the analysis is presented in Theorem 4.1. Computational experiments illustrate analytical derivations and provide insight in the complexity of dynamical processes taking place in the system of nonlinear differential equations with the multiplicative coupling.

It is clear that the variety of solutions to a system of Riccati equations coupled with a multiplicative term is much wider than the first order and the second order solitary solutions. This paper presents the mathematical framework for the construction of a solution to a system of nonlinear differential equations. This framework is based on the inverse balancing of system parameters and the application of the generalized differential operator method for the construction of solitary solutions. Similar approach could be used for the construction of more complex closed form solutions to the system of Riccati equations coupled with a multiplicative term what remains a definite objective of the future research.

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Appendix A. The derivation of sufficient conditions for (4.9)

Computer algebra can be used to prove that:

$$W_{\hat{y}} = A_6(s,t)\eta^6 + A_4(s,t)\eta^4 + A_2(s,t)\eta^2 + A_0(s,t), \tag{A.1}$$

where $A_k(s,t)$ are polynomials in s and t. We seek such to find the roots η_0 of (A.1) such that they do not depend on s, t.

Noting that:

$$A_6 = K^3, \quad K := \sqrt[3]{\frac{1}{2160}}\left(s^2 a_2 + (tb_2 + a_1)\, s + a_0\right), \tag{A.2}$$

(A.1) can be expressed as:

$$K^3\eta^6 + \left(-K^3{\eta_0}^2 + KB_2\right)\eta^4 + \left(-B_2{\eta_0}^2 + B_0\right)\eta^2 - B_0{\eta_0}^2 = 0, \tag{A.3}$$

where B_0, B_2 are polynomials in s, t. It can be observed that there exist such s_0, t_0, that $K\Big|_{s=s_0,t=t_0} = 0$:

$$t_0 = -\frac{a_2 s_0^2 + a_1 s_0 + a_0}{b_2 s_0}, s_0 \in \mathbb{R}. \tag{A.4}$$

Inserting (A.4) into (A.3) yields the condition:

$$\eta_0^2 = \text{const} = -\frac{A_0\,(s_0, t_0)}{A_2\,(s_0, t_0)}. \tag{A.5}$$

The (A.5) can be expressed explicitly:

$$\frac{\left(57\,a_0 a_1 a_2 - 69\,a_0 a_2 b_1 - 16\,a_1{}^3 + 24\,a_1{}^2 b_1 - 24\,a_1 b_0 b_2 - 6\,a_1 b_1{}^2 + 12\,b_0 b_1 b_2 - b_1{}^3\right) s_0 + 15 a_0 \left(3 a_0 a_2 - a_1{}^2 + a_1 b_1 + 3\,b_0 b_2 - b_1{}^2\right)}{\left(30\,a_1 - 15\,b_1\right) s_0 + 45 a_0} = -\eta_0^2. \tag{A.6}$$

Equation (A.6) can be written more compactly:

$$\eta_0^2 = -\frac{\alpha_1 s_0 + \beta_1}{\alpha_2 s_0 + \beta_2}. \tag{A.7}$$

Analogous computations can be performed for the determinant $W_{\hat{z}}$, yielding:

$$\frac{\left(57\,b_0 b_1 b_2 - 69\,b_0 b_2 a_1 - 16\,b_1{}^3 + 24\,b_1{}^2 a_1 - 24\,b_1 a_0 a_2 - 6\,b_1 a_1{}^2 + 12\,a_0 a_1 a_2 - a_1{}^3\right) t_0 + 15\,b_0 \left(3\,a_0 a_2 - a_1{}^2 + a_1 b_1 + 3\,b_0 b_2 - b_1{}^2\right)}{\left(30 b_1 - 15 a_1\right) t_0 + 45 b_0} = -\eta_0^2. \tag{A.8}$$

Denoting the coefficients as in (A.7) yields:

$$\eta_0^2 = -\frac{\alpha_1' t_0 + \beta_1'}{\alpha_2' t_0 + \beta_2'}. \tag{A.9}$$

Note that (A.7) and (A.9) yield:

$$\eta_0^2 = -\frac{\alpha_1 s_0 + \beta_1}{\alpha_2 s_0 + \beta_2} = -\frac{\alpha_1' t_0 + \beta_1'}{\alpha_2' t_0 + \beta_2'}, \tag{A.10}$$

and the fractions do not depend on s_0 and t_0.

The fractions in (A.7) and (A.9) do not depend on s_0, t_0 and (A.10) is satisfied if

$$R := 9a_0 a_1 a_2 + 9b_0 b_1 b_2 - 18 a_0 a_2 b_1 - 18 b_0 b_2 a_1 + 3 a_1 b_1^2 + 3 b_1 a_1^2 - 2a_1^3 - 2b_1^3 = 0, \tag{A.11}$$

because

$$\alpha_1 \beta_2 - \alpha_2 \beta_1 = 135 a_0 R; \quad \alpha_1' \beta_2' - \alpha_2' \beta_1' = 135 b_0 R. \tag{A.12}$$

If (A.11) holds, then:

$$\eta_0^2 = \frac{a_1^2 - a_1 b_1 + b_1^2}{3} - a_0 a_2 - b_0 b_2. \tag{A.13}$$