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Kink solitary solutions to generalized Riccati equations with polynomial coefficients

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ABSTRACT

Inverse and direct balancing methods for the construction of kink solitary solutions to generalized Riccati equations with polynomial coefficients are presented in this paper. Necessary and sufficient conditions for the existence of kink solitary solutions are derived in terms of the equation's parameters and initial conditions. The proposed technique enables a straightforward derivation of parameters of solitary solutions. Computational experiments are used to demonstrate the efficiency of the proposed analytical approach.

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1. Introduction

The Riccati differential equation

$$y'_x = a_2(x)y^2 + a_1(x)y + a_0(x), \tag{1}$$

where $a_0 \neq 0, a_1$ and $a_2 \neq 0$ are arbitrary functions, is one of the most important equations in mathematical physics [4,23,28]. Though first defined in the 18th century [29], the Riccati equation remains one the foremost problems in differential equations research today (because it cannot be solved in quadratures [10]).

A number of novel methods have been developed in recent years to construct solutions to (1). Solutions to Riccati equations with constant and variable coefficients are obtained using the iterative reproducing kernel Hilbert spaces method in [30]. The number of polynomial solutions to the Riccati equation with polynomial coefficients is considered in [8].

The operation matrix method based on Bernstein polynomials is used to construct numerical solutions to the Riccati equation in [26]. The partial fraction decomposition is used to obtain rational solutions to

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quadratic Riccati equations with rational coefficients in [7]. Quarternionic Riccati equations are applied to study quarternionic Schrödinger equations in $(1+1)$ dimensions [6].

There exists a number of generalizations of the classical Riccati equation. Boundary problems on the matrix Riccati differential equation are considered in [13]. The integrability of Abel differential equations

$$y'_x = a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x), \tag{2}$$

where $a_k(x)$, $k = 0, \ldots, 3$ are meromorphic functions is studied in [9]. Implicit solutions to (2) are constructed using transformations of known cases of solvable Abel equations in [16]. Yamaleev considers generalized Riccati equations of the form [32]:

$$y'_x = a_n(x)y^n + \cdots + a_1(x)y + a_0(x), \quad n \in \mathbb{N}. \tag{3}$$

Special cases of equation (3) (sometimes also called the generalized Abel equation) have been considered in literature. Discussions on (3) in the case $n = 3$ can be found in [14,24]. The case of T-periodic coefficients is considered in [2]. An estimate of the number of limit cycles in (3) is given in [1]. The number of periodic solutions to (3) is studied in [25].

The main objective of this paper is to provide an analytical framework for the construction of solutions

$$y = \frac{\alpha_1 x - \alpha_0}{\beta_1 x - \beta_0}, \quad \alpha_1, \alpha_0, \beta_1, \beta_0 \in \mathbb{R}; \tag{4}$$

to (3) in the following special case:

$$\left(\sum_{k=0}^{r} b_k x^k\right) y_x^{(m)} = \sum_{l=0}^{n} a_l y^l, \tag{5}$$

where b_k, $a_l \in \mathbb{R}$; $k = 0, \ldots, r$; $l = 0, \ldots, n$. The objective of this paper is to derive conditions of existence for (4) in (5) in terms of the equation and the solution parameters.

2. Preliminaries and motivation

It is well known that the Riccati equation with constant coefficients

$$u'_z = a_2 u^2 + a_1 u + a_0, \quad u(z_0) = u_0; \quad a_2, a_1, a_0 \in \mathbb{R}, \tag{6}$$

admits only kink solitary solutions of the following form [19,27]:

$$u = u_2 \frac{\exp\left(a_2 \left(u_1 - u_2\right)\left(z - z_0\right)\right) - \frac{u_1(u_0 - u_2)}{u_2(u_0 - u_1)}}{\exp\left(a_2 \left(u_1 - u_2\right)\left(z - z_0\right)\right) - \frac{u_0 - u_2}{u_0 - u_1}}, \tag{7}$$

where u_1, u_2 are roots of the polynomial $a_2u^2 + a_1u + a_0$.

However, kink solitary solutions can have analytical expressions that are different from (7), while still satisfying equations that are similar in form to (6). The analytic expression of kink solitary solutions can be written more generally as:

$$u = \frac{\alpha_1 f(z) - \alpha_0}{\beta_1 f(z) - \beta_0}; \quad \alpha_1, \alpha_0, \beta_1, \beta_0 \in \mathbb{R}, \quad \alpha_1\beta_0 - \alpha_0\beta_1 \neq 0, \tag{8}$$

where $f(z)$ is an arbitrary invertible analytic function. It is clear that (8) is a generalization of (7). Based on the well-known Exp-function method [3,18], the following substitution can be introduced:

$$x = \frac{1}{\eta} \ln z, \quad z = \exp(\eta x); \quad \eta \neq 0. \tag{9}$$

Denoting $y = y(x) = u(\exp(\eta x))$ transforms (8) into

$$y = \frac{\alpha_1 x - \alpha_0}{\beta_1 x - \beta_0}. \tag{10}$$

It is known from numerous previous research [12,20,22,27,31] that (10) does satisfy the differential equation

$$x y'_x = c_2 y^2 + c_1 y + c_0, \quad y(x_0) = y_0; \quad c_2, c_1, c_0 \in \mathbb{R}. \tag{11}$$

Thus it can be concluded that the equation (11) is a generalized form of (6) and in turn can be used to generate numerous kink solitary solutions.

However, substitution (9) is not the only option that generates kink solitary solutions – this is demonstrated in Example 2.1.

Example 2.1. Let us consider the following differential equation:

$$3x y'_x = 2y^2 - y - 1, \quad y(x_0) = y_0. \tag{12}$$

The general solution to (12) reads:

$$y = -\frac{1}{2} \frac{x + 2x_0 \dfrac{y_0 + 1/2}{y_0 - 1}}{x - x_0 \dfrac{y_0 + 1/2}{y_0 - 1}}. \tag{13}$$

A number of kink solitary solutions can be constructed from (13).

Exponential kink solitary solution. Applying the substitution

$$x = \exp(\eta z), \quad z = \frac{1}{\eta} \ln x; \quad \eta \in \mathbb{R}, \tag{14}$$

yields the solitary solution:

$$u = u(z) = y(\exp(\eta z)) = -\frac{1}{2} \frac{\exp(\eta(z - z_0)) + \frac{u_0 + 1/2}{u_0 - 1}}{\exp(\eta(z - z_0)) - \frac{u_0 + 1/2}{u_0 - 1}}, \tag{15}$$

which satisfies the Riccati equation:

$$u'_z = \frac{2\eta}{3}(u - 1)\left(u + \frac{1}{2}\right), \quad u(z_0) = u_0. \tag{16}$$

Radical kink solitary solution. The solution and equation pair (equations (13), (12)) can be used to obtain solitary solutions that are different from the traditional form (7). Applying the substitution

$$x = \sqrt{6z - 1}, \quad z = \frac{x^2 + 1}{6}, \tag{17}$$

yields the solitary solution:

$$u = u(z) = y\left(\sqrt{6z-1}\right) = \frac{\sqrt{\frac{6z-1}{6z_0-1}} + 2\frac{u_0+1/2}{u_0-1}}{\sqrt{\frac{6z-1}{6z_0-1}} - \frac{u_0+1/2}{u_0-1}}, \tag{18}$$

which satisfies the generalized Riccati equation:

$$(6z-1)\,u'_z = 2\,(u-1)\left(u+\frac{1}{2}\right), \quad u(z_0) = u_0. \tag{19}$$

The above example demonstrates that the construction of solutions to (11) allows to consider a wide range of kink solitary solutions in a variety of generalized Riccati equations.

In the remainder of this paper, the following general case of (11) is considered:

$$\left(\sum_{k=0}^{r} b_k x^k\right) y_x^{(m)} = \sum_{l=0}^{n} a_l y^l; \quad r, m, n \in \mathbb{Z}_0, \quad a_n = 1, b_r \neq 0. \tag{20}$$

Conditions for the existence of kink solitary solutions (10) to (20) are derived in the subsequent sections.

3. Inverse balancing – necessary existence conditions

Inverse balancing method has been successfully used to determine if a given differential equation can accept solitary solutions [15,22]. The method is based on the assumption that the parameters of solution (10) are given and the parameters of differential equation (20) are unknown. The solution is inserted into the differential equation and a system of linear equations for the determination of the equation's parameters is constructed. The conditions under which the linear system accepts nontrivial solutions coincide with the necessary conditions for existence of solution (10) in (20).

3.1. Balancing the order of equation, coefficients and the nonlinearity

The orders of the equation (20) (m), the coefficients (r) and the nonlinear terms (n) must be balanced if (10) is to be a solution to (20). Using the inverse balancing method, it is assumed that the parameters of the solution α_1, α_0, β_1, β_0 are given and the parameters of the equation $m, r, n, b_0, \dots, b_r, a_0, \dots, a_n$ are unknown.

Note that the jth derivative of (10) reads:

$$y_x^{(j)} = \frac{(-1)^j\, j!\Theta\beta_1^{j-1}}{(\beta_1 x - \beta_0)^{j+1}}, \quad j = 1, \dots \tag{21}$$

where $\Theta = \alpha_1\beta_0 - \alpha_0\beta_1 \neq 0$. Inserting the expressions (10) and (21) into (20) yields:

$$\frac{(-1)^m\, m!\Theta\beta^{m-1}}{(\beta_1 x - \beta_0)^{m+1}}\left(\sum_{k=0}^{r} b_k x^k\right) = \frac{1}{(\beta_1 x - \beta_0)^n}\sum_{l=0}^{n} a_l\,(\alpha_1 - a_0)^l\,(\beta_1 x - \beta_0)^{n-l}. \tag{22}$$

Equation (22) can hold for all $x \in \mathbb{R}$ only if the degrees of the numerators and denominators on both sides are equal. This can only be true for $a_n \neq 0$ when $n = m + 1$. Furthermore, because $b_r \neq 0$, the condition $0 \leq r \leq m + 1$ must be satisfied – or else the condition $b_{r+1} = \cdots = b_{m+2} = 0$ must hold. Thus, from the structure of (22) it can be concluded that the following balancing conditions are necessary to ensure the existence of nontrivial solutions of form (10) to (20):

$$n = m + 1; \tag{23}$$

$$0 \leq r \leq m + 1. \tag{24}$$

Balancing conditions (23), (24) indicate that (20) can admit the kink solitary solution (10) if it is has the following form:

$$\left(\sum_{k=0}^{r} b_k x^k\right) y_x^{(m)} = \sum_{l=0}^{m+1} a_l y^l; \quad m \in \mathbb{N}, \quad 0 \leq r \leq m + 1. \tag{25}$$

3.2. Determination of the equation's coefficients from the solution parameters

Let the conditions (23) and (24) do hold. Multiplying both sides of (22) by $(\beta_1 x - \beta_0)^{m+1}$ results in:

$$\left(\sum_{k=0}^{r} b_k x^k\right) (-1)^m m!\Theta\beta_1^{m-1} = \sum_{l=0}^{m+1} a_l (\alpha_1 x - \alpha_0)^l (\beta_1 x - \beta_0)^{m-l+1}. \tag{26}$$

Using (26), the following system of $m + 2$ linear equations with respect to the coefficients $b_0, \ldots, b_r$, $a_0, \ldots, a_{m+1}$ can be constructed:

$$\frac{\mathrm{d}^j}{\mathrm{d}\, x^j}\left(\left(\sum_{k=0}^{r} b_k x^k\right) (-1)^m m!\Theta\beta_1^{m-1} - \sum_{l=0}^{m+1} a_l (\alpha_1 x - \alpha_0)^l (\beta_1 x - \beta_0)^{m-l+1}\right)\Bigg|_{x=0} = 0, \tag{27}$$

where $j = 0, 1, \ldots, m+1$. Because the system has $m+1$ linear equations and $2m+1$ unknowns, depending on the rank of the system's matrix, some parameters can be chosen freely and the others computed from (27).

Example 3.1. Let us consider the following differential equation:

$$\left(b_2 x^2 + b_1 x + b_0\right) y'_x = y^2 + a_1 y + a_0. \tag{28}$$

Note $m = 1$, $n = r = 2$, thus the balancing conditions (23), (24) are satisfied. Assuming that the parameters α_1, α_0, β_1, β_0 of the solution (10) are known, inserting (10) into (28) and simplifying yields a special case of the linear system (27):

$$\begin{aligned}&\left(-2\beta_1^2 a_0 - 2\alpha_1\beta_1 a_1 - 2\Theta b_2 - 2\alpha_1^2\right) x^2 + \left(2\beta_0\beta_1 a_0 + (\alpha_0\beta_1 + \alpha_1\beta_0) a_1\right.\\ &\left.- \Theta b_1 + 2\alpha_0\alpha_1\right)x - \beta_0^2 a_0 - \alpha_0\beta_0 a_1 - \Theta b_0 - \alpha_0^2 = 0.\end{aligned} \tag{29}$$

Thus (29) yields the linear system with respect to a_0, a_1, b_0, b_1, b_2:

$$\begin{aligned}&-2\beta_1^2 a_0 - 2\alpha_1\beta_1 a_1 - 2\Theta b_2 - 2\alpha_1^2 = 0;\\ &2\beta_0\beta_1 a_0 + (\alpha_0\beta_1 + \alpha_1\beta_0) a_1 - \Theta b_1 + 2\alpha_0\alpha_1 = 0;\\ &-\beta_0^2 a_0 - \alpha_0\beta_0 a_1 - \Theta b_0 - \alpha_0^2 = 0.\end{aligned} \tag{30}$$

Letting a_0, a_1 be free parameters, the solution for b_0, b_1, b_2 reads:

$$\begin{aligned}&b_0 = -\frac{1}{\Theta}\left(a_0\beta_0^2 + a_1\alpha_0\beta_0 + \alpha_0^2\right), \quad b_2 = -\frac{1}{\Theta}\left(a_0\beta_1^2 + a_1\alpha_1\beta_1 + \alpha_1^2\right);\\ &b_1 = \frac{1}{\Theta}\left(2a_0\beta_0\beta_1 + a_1\alpha_0\beta_1 + a_1\alpha_1\beta_0 + 2\alpha_0\alpha_1\right), \quad a_0, a_1 \in \mathbb{R}.\end{aligned} \tag{31}$$

The solution (31) demonstrates the infinite set of differential equations that admit the kink solitary solution (10). Note that while the equations with coefficients determined by (31) do admit the solutions (10), the inverse balancing method does not give an algorithm for the construction of a solution for a given equation (28). This problem is addressed in the subsequent sections.

4. Direct balancing – construction of kink solitary solutions

Direct balancing is a technique used to compute the parameters of the kink solitary solution (10) by inserting it into the differential equation (20) and solving for the solution parameters. Note that this method can only be applied when the resulting equations are linear with respect to the solution parameters α_1, α_0, β_1, β_0. While there are a number of ansatz-based methods used in a similar way, they have been heavily criticized for yielding incorrect solutions precisely because the equations for the solution are nonlinear and their complete solutions often cannot be obtained [3,11,12,17,20].

4.1. Parameter equations for the kink solitary solution

The generalized Riccati equation (25) can be rewritten in the following form:

$$b_r \prod_{k=1}^{r} (x - x_k)\, y_x^{(m)} = \prod_{l=1}^{m+1} (y - y_l)\,, \tag{32}$$

where $0 \le r \le m+1$, $b_r \neq 0$; x_k, $k = 1, \ldots, r$ and y_l, $l = 1, \ldots, m+1$ are the roots of polynomials $\sum_{k=0}^{r} b_r x^k$ and $\sum_{l=0}^{m+1} a_l y^l$ respectively.

Inserting (10) and (21) into (32) yields:

$$b_r \prod_{k=1}^{r} (x - x_k) \frac{(-1)^m\, m!\Theta\beta_1^{m-1}}{(\beta_1 x - \beta_0)^{m+1}} = \prod_{l=1}^{m+1} \left(\frac{\alpha_1 x - \alpha_0}{\beta_1 x - \beta_0} - y_l \right). \tag{33}$$

Canceling like terms results in:

$$b_r (-1)^m\, m!\Theta\beta_1^{m-1} \prod_{k=1}^{r} (x - x_k) = \prod_{l=1}^{m+1} \left((\alpha_1 - y_l\beta_1)\, x - (\alpha_0 - y_l\beta_0) \right). \tag{34}$$

Let us consider any $(m+1)$st order permutation $\tau \in \mathcal{S}_{m+1}$:

$$\tau := \begin{pmatrix} 1 & 2 & \ldots & m+1 \\ \tau(1) & \tau(2) & \ldots & \tau(m+1) \end{pmatrix}. \tag{35}$$

In the remainder of the text, the shorter notation $\tau := (\tau(1), \ldots, \tau(m+1))$ will be used.

Note that $r \le m$, thus r terms from the right side of (34) with indices $\tau(1), \ldots, \tau(r)$ can be chosen and (34) is rearranged as follows:

$$\begin{aligned} b_r (-1)^m\, m!\Theta\beta_1^{m-1} \prod_{k=1}^{r} (x - x_k) &= \left(\prod_{l=1}^{r} \left(\alpha_1 - y_{\tau(l)}\beta_1\right) \right) \left(\prod_{l=1}^{r} \left(x - \frac{\alpha_0 - y_{\tau(l)}\beta_0}{\alpha_1 - y_{\tau(l)}\beta_1} \right) \right) \\ &\quad \times \left(\prod_{j=r+1}^{m+1} \left(\left(\alpha_1 - y_{\tau(j)}\beta_1\right) x - \left(\alpha_0 - y_{\tau(j)}\beta_0\right) \right) \right). \end{aligned} \tag{36}$$

By considering the coefficients on the left and right side of (36), it can concluded that (10) satisfies (32) if the following system of equations holds true:

$$x_k = \frac{\alpha_0 - y_{\tau(k)}\beta_0}{\alpha_1 - y_{\tau(k)}\beta_1}, \quad k = 1, \dots, r; \tag{37}$$

$$\alpha_1 - y_{\tau(j)}\beta_1 = 0, \quad j = r+1, \dots, m+1; \tag{38}$$

$$b_r \left(-1\right)^m m!\Theta\beta_1^{m-1} = \left(-1\right)^{m-r+1} \left(\prod_{l=1}^{r+1} \left(\alpha_1 - y_{\tau(l)}\beta_1\right)\right) \left(\prod_{j=r+1}^{m+1} \left(\alpha_0 - y_{\tau(j)}\beta_0\right)\right). \tag{39}$$

Rearranging (37) and (38) yields $m+1$ linear equations with respect to α_1, α_0, β_1, β_0:

$$x_k\alpha_1 - \alpha_0 - x_k y_{\tau(k)}\beta_1 + y_{\tau(k)}\beta_0 = 0, \quad k = 1, \dots, r; \tag{40}$$

$$\alpha_1 - y_{\tau(l)}\beta_1 = 0, \quad l = r+1, \dots, m+1. \tag{41}$$

The coefficient b_r must satisfy equation (39) if (10) satisfies (32). Equation (39) can be rearranged in the following form:

$$b_r = \frac{\left(-1\right)^{1-r}}{m!\Theta\beta_1^{m-1}} \left(\prod_{l=1}^{r} \left(\alpha_1 - y_{\tau(l)}\beta_1\right)\right) \left(\prod_{j=r+1}^{m+1} \left(\alpha_0 - y_{\tau(j)}\beta_0\right)\right). \tag{42}$$

Note that (42) should not be used to determine the solution coefficients α_1, α_0, β_1, β_0. Instead, the validity of equation (42) must be tested after obtaining the parameters of the solution.

4.2. Necessary and sufficient existence conditions

Equations (40) and (41) define $(m+1)!$ linear systems that depend on the chosen permutation $\tau \in \mathcal{S}_{m+1}$. The systems can be written in the matrix form:

$$\boldsymbol{A}\left(\tau\right)\boldsymbol{p}(\tau) = \boldsymbol{0}, \tag{43}$$

where

$$\boldsymbol{A}\left(\tau\right) = \begin{bmatrix} x_1 & -1 & -x_1 y_{\tau(1)} & y_{\tau(1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_r & -1 & -x_r y_{\tau(r)} & y_{\tau(r)} \\ 1 & 0 & -y_{\tau(r+1)} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & -y_{\tau(m+1)} & 0 \end{bmatrix}; \tag{44}$$

and

$$\boldsymbol{p}(\tau) = \left[\alpha_1(\tau) \quad \alpha_0(\tau) \quad \beta_1(\tau) \quad \beta_0(\tau)\right]^T. \tag{45}$$

The solution parameters depend on τ indirectly – they are different for different permutations τ in the general case.

Theorem 4.1. *The generalized Riccati equation (5) admits the solution (10) if and only if (23), (24), (42) hold true and $\boldsymbol{A}\left(\tau\right)$ satisfies:*

$$\operatorname{rank}\left(\boldsymbol{A}\left(\tau\right)\right) < 4. \tag{46}$$

Proof. The balancing conditions (23), (24) ensure the necessity.

If (42) holds true, the structure of matrix $\boldsymbol{A}(\tau)$ implies that (43) has nontrivial solutions only if (46) holds. Thus (42) and (46) are sufficient existence conditions. $\square$

4.3. Initial conditions for solitary solutions

If the conditions of Theorem 4.1 are satisfied, solutions to (5) of the form (10) do exist. However, they do not hold true for all initial conditions. Let us consider the initial conditions for (5) of order m:

$$y(x_0) = y_0; \tag{47}$$

$$y_x^{(k)}\Big|_{x=x_0} = y_0^{(k)}; k = 1, 2, \ldots, m. \tag{48}$$

When (10) satisfies the initial conditions (47) and (48), x_0 can be expressed as:

$$x_0 = \frac{\alpha_0 - \beta_0 y_0}{\alpha_1 - \beta_1 y_0}. \tag{49}$$

Equation (21) together with (48) and (49) yields:

$$y_0^{(k)} = \frac{-k!\beta_1^{k-1}}{\Theta^k}\left(\alpha_1 - \beta_0 y_0\right)^{k+1}. \tag{50}$$

The initial conditions must satisfy (49) and (50) if (10) is a solution to the initial value problem on (5).

5. Computational experiments

5.1. The Abel equation of the first kind

Let us consider the following generalized Riccati equation:

$$b_3(x-1)^2(x+1)y''_{xx} = (y-1)^2(y+1), \quad y(x_0) = y_0, \ y'\Big|_{x=x_0} = y'_0; \quad b_3 \in \mathbb{R}. \tag{51}$$

Because $r = 3$, $m = 2$, $n = 3$, equation (51) satisfies balancing conditions (23), (24).

Let us denote $x_1 = x_2 = 1$, $x_3 = -1$ and $y_1 = y_2 = 1$, $y_3 = -1$. For any permutation $\tau \in \mathcal{S}_3$, matrix (44) reads:

$$\boldsymbol{A}(\tau) = \begin{bmatrix} 1 & -1 & -y_{\tau(1)} & y_{\tau(1)} \\ 1 & -1 & -y_{\tau(2)} & y_{\tau(2)} \\ -1 & -1 & y_{\tau(3)} & y_{\tau(3)} \end{bmatrix}. \tag{52}$$

Case 1. Let $\tau = (1, 2, 3)$ or $\tau = (2, 1, 3)$. Then $\operatorname{rank}(\boldsymbol{A}(\tau)) = 2$ and condition (46) is satisfied. The solution to (43) reads:

$$\alpha_1 = -\beta_0, \quad \alpha_0 = -\beta_1, \quad \beta_0, \beta_1 \in \mathbb{R}. \tag{53}$$

Parameters (53) yield the solitary solution:

$$y = \frac{\beta_1 - \beta_0 x}{\beta_1 x - \beta_0}. \tag{54}$$

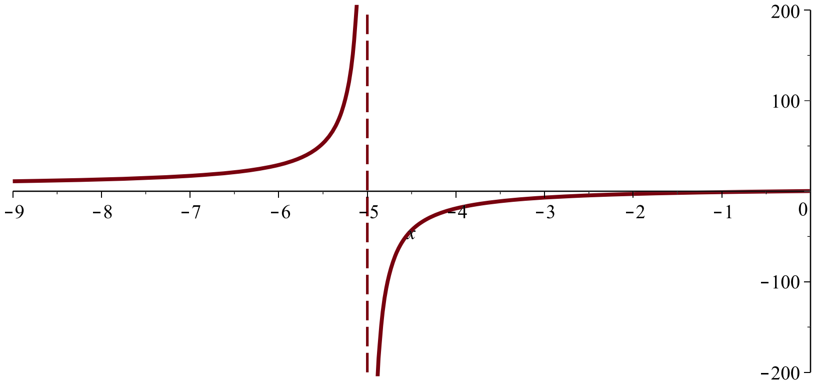


Fig. 1. Solitary solution (54) to (51) with $\beta_0 = -1$, $\beta_1 = 5$. The dashed line denotes the singularity of the solution at $x = -5$.

However, (54) satisfies (51) only if b_3 satisfies (42):

$$b_3 = \frac{\beta_1 + \beta_0}{2\beta_1}. \tag{55}$$

Thus (54) is a solution to

$$\frac{\beta_1 + \beta_0}{2\beta_1} (x-1)^2 (x+1) y''_{xx} = (y-1)^2 (y+1), \quad y(x_0) = y_0, \ y'\Big|_{x=x_0} = y'_0, \tag{56}$$

where $\beta_0, \beta_1 \in \mathbb{R}$. The solution (54) is depicted in Fig. 1.
Note that solution (54) does not hold for all initial conditions x_0, y_0, y'_0. The initial conditions of (56) yield:

$$y_0 = \frac{\beta_1 - \beta_0 x_0}{\beta_1 x_0 - \beta_0}, \quad y'_0 = \frac{\beta_0^2 - \beta_1^2}{(\beta_1 x_0 - \beta_0)^2}. \tag{57}$$

Eliminating x_0 from (57) as

$$x_0 = \frac{\beta_0 y_0 + \beta_1}{\beta_1 y_0 + \beta_0}, \tag{58}$$

results in the initial condition constraint:

$$y'_0 = \frac{(\beta_1 y_0 + \beta_0)^2}{\beta_0^2 - \beta_1^2}. \tag{59}$$

The initial condition constraint (59) can be verified by means of a computational experiment. Using a time-forward constant-step numerical ODE integrator, approximate solutions to (56) with initial conditions x_0, y_0, y'_0 can be constructed. Let us denote the approximate solution to (56) obtained by the classical RK4 method [5] as $\widehat{y}(x; x_0, y_0, y'_0)$. The difference between the numerical solution $\widehat{y}$ and (54) can be defined as:

$$\Delta(y_0, y'_0) = \sum_{j=1}^{N} \left| y(jh; x_0, y_0, y'_0) - \widehat{y}(jh; x_0, y_0, y'_0) \right|, \tag{60}$$

where N is the number of time-forward steps; h is the stepsize and x_0 satisfies (58).

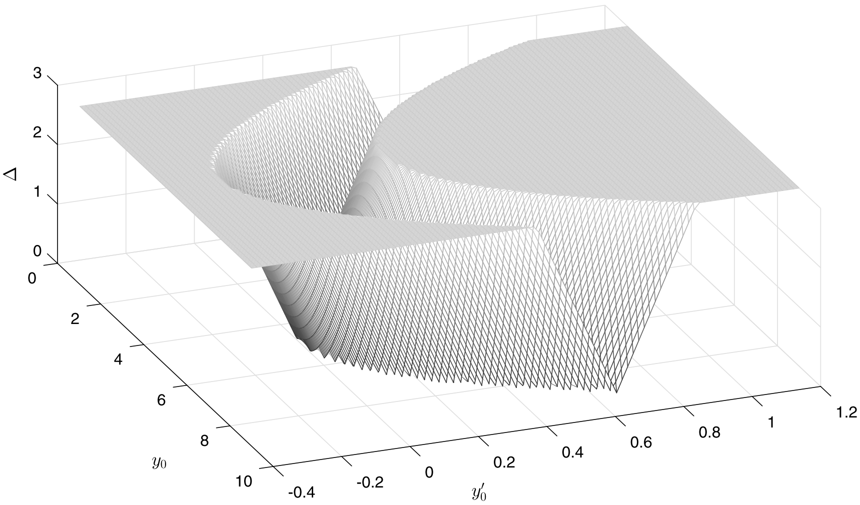


Fig. 2. The difference $\Delta\left(y_0, y_0'\right)$ for $\beta_0 = 5$, $\beta_1 = -1$ as $1 \le y_0 \le 9$ and $-0.4 \le y_0' \le 1.2$. The step-size $h = 0.01$; the difference is estimated over $N = 50$ time-forward steps. Values higher than 3 have been truncated to 3 for clarity.

The plot of Δ for (56) is depicted in Fig. 2. Note that Δ is almost equal to zero on the parabola defined by the constraint (59).

Case 2. If $\tau \ne (1,2,3),(2,1,3)$ then $\operatorname{rank}(\boldsymbol{A}(\tau)) = 1$ and condition (46) is satisfied. In this case the solution to (43) reads:

$$\alpha_0 = \alpha_1 = \beta_0 = \beta_1. \tag{61}$$

However, $\Theta = 0$ and (42) is invalid. In this case, the trivial non-solitary solutions $y = 1$, $y = -1$ are admitted for any $b_3 \in \mathbb{R}$.

5.2. Third order generalized Riccati equation

Let us consider the following third-order generalized Riccati equation:

$$b_3\left(x^4 - x^3 - 22x^2 + 16x + 96\right) y'''_{xxx} = y^4 + \frac{159}{41}y^3 - \frac{794}{41}y^2 - \frac{1776}{41}y - \frac{864}{41}, \tag{62}$$

where $b_3 \in \mathbb{R}$. The initial conditions read:

$$y\left(x_0\right) = y_0, \quad y'_x\Big|_{x=x_0} = y'_0, \quad y''_{xx}\Big|_{x=x_0} = y''_0; \quad x_0, y_0, y'_0, y''_0 \in \mathbb{R}. \tag{63}$$

Note that $r = 4$, $m = 3$, $n = 4$, thus (62) satisfies the necessary solitary solution existence conditions (23), (24).

Equation (62) can be rewritten as:

$$b_3\left(x-4\right)\left(x+4\right)\left(x+2\right)\left(x-3\right) y'''_{xxx} = \left(y-4\right)\left(y + \frac{36}{41}\right)\left(y+1\right)\left(y+6\right). \tag{64}$$

The roots x_k, y_k, $k = 1, \ldots, 4$ are enumerated as follows:

$$x_1 = 4, \quad x_2 = -4, \quad x_3 = -2, \quad x_4 = 3; \tag{65}$$

$$y_1 = 4, \quad y_2 = -\frac{36}{41}, \quad y_3 = -1, \quad y_4 = -6. \tag{66}$$

For any permutation $\tau \in \mathcal{S}_4$, matrix (44) reads:

$$\boldsymbol{A}(\tau) = \begin{bmatrix} 4 & -1 & 4y_{\tau(1)} & y_{\tau(1)} \\ -4 & -1 & -4y_{\tau(2)} & y_{\tau(2)} \\ -2 & -1 & -2y_{\tau(3)} & y_{\tau(3)} \\ 3 & -1 & 3y_{\tau(4)} & y_{\tau(4)} \end{bmatrix}. \tag{67}$$

The four cases of $\tau \in \mathcal{S}_4$ that result in nontrivial solutions to (45) and thus yield kink solitary solutions are given below.

Case 1. The permutation $\tau_1 = (1, 2, 3, 4)$ yields that $\operatorname{rank}(\boldsymbol{A}(\tau_1)) = 3$. The solution to (45) reads:

$$\alpha_1 = -\frac{6}{11}\beta_1, \quad \alpha_0 = -\frac{48}{11}\beta_1, \quad \beta_0 = \frac{38}{11}\beta_1; \quad \beta_1 \in \mathbb{R}. \tag{68}$$

The parameters (68) correspond to the solution

$$y = \frac{6(8-x)}{11x-38}, \tag{69}$$

which satisfies (62) if the coefficient b_3 satisfies (42):

$$b_3 = \frac{1250}{4961}. \tag{70}$$

The solution (69) is pictured in Fig. 3.

It can be observed that initial conditions (63) are satisfied when x_0 is eliminated as:

$$x_0 = \frac{2(19y_0 + 24)}{11y_0 + 6}, \tag{71}$$

and the following constraints hold true:

$$y_0' = -\frac{1}{300}(11y_0 + 6)^2; \tag{72}$$

$$y_0'' = \frac{11}{45000}(11y_0 + 6)^3. \tag{73}$$

To verify (71)–(73), the numerical experiment described in the previous subsection is used. The difference between (69) and the numerical solution to (62) is defined as:

$$\Delta_k\left(y_0, y_0^{(k)}\right) = \sum_{j=1}^{N} \left| y(jh; x_0, y_0, y_0', y_0'') - \widehat{y}(jh; x_0, y_0, y_0', y_0'') \right|, \quad k = 1, 2. \tag{74}$$

The plots of Δ_1, Δ_2 are pictured in Fig. 4. Note that the difference between the numerical solution and (69) are almost equal to zero on the curves defined by (72), (73).

Case 2. Let $\tau_2 = (2, 1, 4, 3)$. Then $\operatorname{rank}(\boldsymbol{A}(\tau_2)) = 3$ and the solution parameters read:

$$\alpha_0 = -32\alpha_1, \quad \beta_1 = -6\alpha_1, \quad \beta_0 = 17\alpha_1; \quad \alpha_1 \in \mathbb{R}. \tag{75}$$

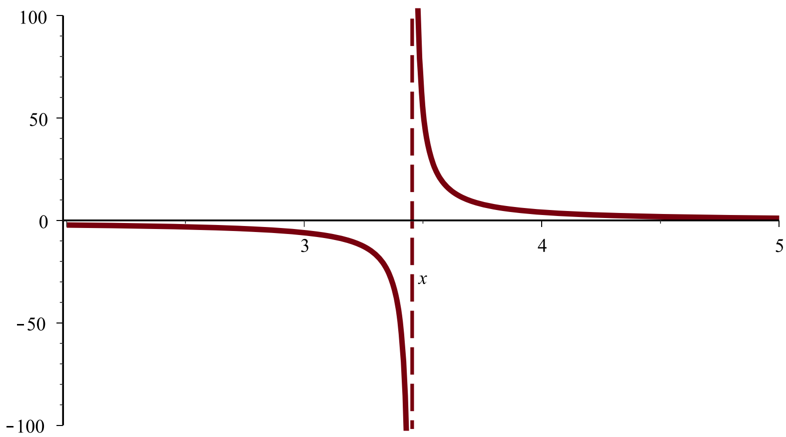


Fig. 3. Solitary solution (69) to (62) for $b_3 = \frac{1250}{4961}$. The dashed line denotes the singularity of the solution at $x = \frac{38}{11}$.

The solitary solution reads:

$$y = -\frac{x+32}{6x+17}. \tag{76}$$

The function (76) is a solution to (62) only if:

$$b_3 = -\frac{4375}{8856}. \tag{77}$$

If (76) is a solution to (62), then the initial conditions must satisfy the following constraints:

$$x_0 = -\frac{17y_0+32}{6y_0+1}, \tag{78}$$

and

$$y_0' = \frac{1}{175}\left(6y_0+1\right)^2; \tag{79}$$

$$y_0'' = \frac{12}{30625}\left(6y_0+1\right)^3. \tag{80}$$

Case 3. The permutation $\tau_3 = (3,4,1,2)$ results in $\operatorname{rank}\left(\boldsymbol{A}\left(\tau_3\right)\right) = 3$ and yields the parameters:

$$\alpha_0 = 0, \quad \beta_1 = -\frac{7}{12}\alpha_1, \quad \beta_0 = \frac{5}{3}\alpha_1; \quad \alpha_1 \in \mathbb{R}. \tag{81}$$

The solution that corresponds to (81) reads:

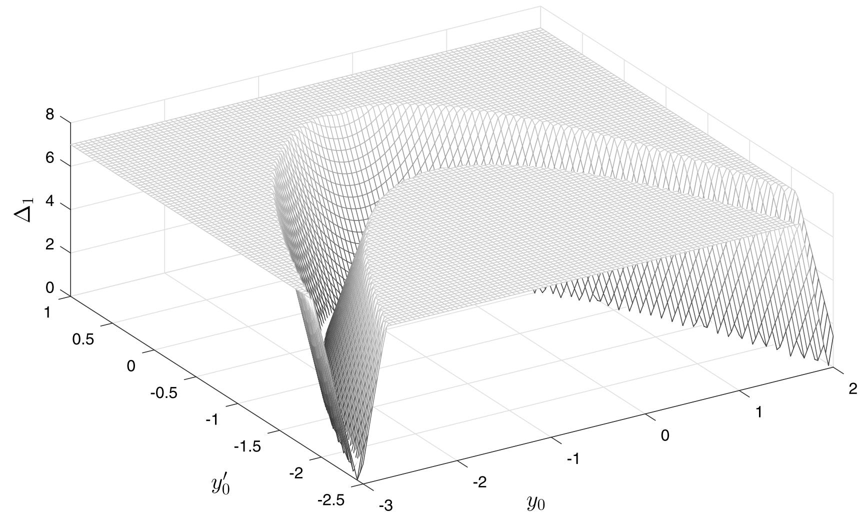
$$y = -\frac{12x}{7x+20}, \tag{82}$$

and satisfies (62) only if:

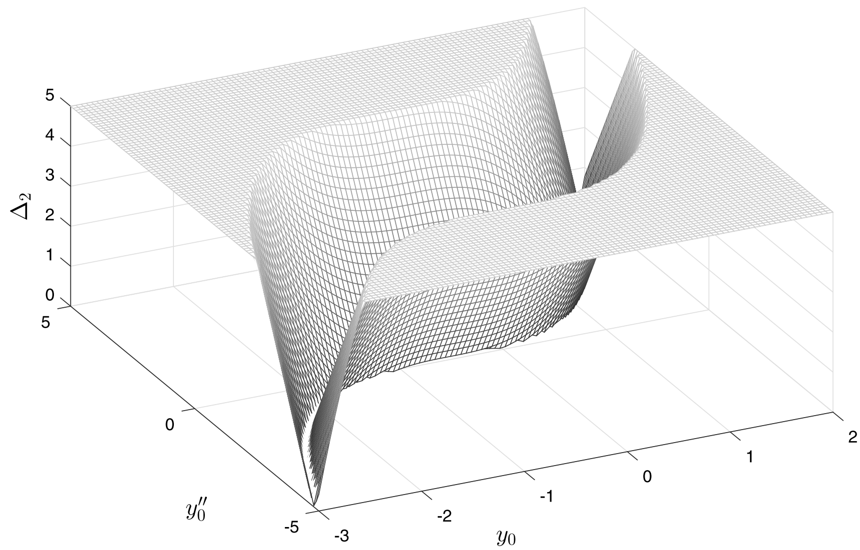
$$b_3 = \frac{1000}{2009}. \tag{83}$$

The initial conditions (63) must satisfy the relation

$$x_0 = -\frac{20y_0}{7y_0+12}, \tag{84}$$



(a)



(b)

Fig. 4. The difference $\Delta_1(y_0, y_0')$ (part (a)); the difference $\Delta_2(y_0, y_0'')$ (part (b)) for $-3 \le y_0 \le 2$, $-2.5 \le y_0' \le 1$, $-5 \le y_0'' \le 5$. In (a), x_0 and y_0'' satisfy (71) and (73) respectively. In (b), x_0 and y_0' satisfy (71) and (72). The step-size $h = 0.01$; the difference is estimated over $N = 50$ time-forward steps. Values higher than 5 and 7 have been truncated to 5 and 7 in (a) and (b) respectively.

as well as:

$$y_0' = -\frac{1}{240}(7y_0 + 12)^2; \tag{85}$$

$$y_0'' = \frac{7}{28800}(7y_0 + 12)^3. \tag{86}$$

Case 4. Letting $\tau_4 = (4, 3, 2, 1)$ results in $\operatorname{rank}(\boldsymbol{A}(\tau_4)) = 3$ and the parameters:

$$\alpha_1 = -\frac{4}{3}\beta_1, \quad \alpha_0 = -\frac{32}{15}\beta_1, \quad \beta_0 = \frac{52}{15}\beta_1; \quad \beta_1 \in \mathbb{R}. \tag{87}$$

The parameters (87) yield the solution:

$$y = \frac{4\,(8-5x)}{15x-52}, \tag{88}$$

which satisfies (62) if b_3 reads:

$$b_3 = -\frac{280}{1107}. \tag{89}$$

The initial conditions that correspond to (88) read:

$$x_0 = \frac{4\,(13y_0+8)}{5\,(3y_0+4)}; \tag{90}$$

$$y_0' = \frac{5}{112}\,(3y_0+4)^2\,; \tag{91}$$

$$y_0'' = \frac{75}{6272}\,(3y_0+4)^3\,. \tag{92}$$

Permutations τ that do not correspond to the cases listed above result in $\operatorname{rank}(\boldsymbol{A}(\tau)) = 4$ and do not satisfy Theorem 4.1.

6. Concluding remarks

It is shown that the generalized Riccati equation with polynomial coefficients (5) admits kink solitary solutions (10). Necessary and sufficient existence conditions have been derived in terms of the equation's parameters using inverse and direct balancing methods. Initial conditions of the Cauchy problem on (5) must satisfy the derived constraints in order to admit the kink solitary solution (10). The efficiency of the inverse and direct balancing methods in constructing kink solitary solutions to (5) is demonstrated by computational experiments.

Note that the direct balancing method can only be used to construct kink solitary solutions. It is important to observe that the algebraic equations for the determination of the solution parameters become nonlinear for higher-order solitary solutions. It is well-known that in the latter case the higher-order solitary solutions obtained by the direct balancing or similar ansatz methods may produce wrong results [3,12,19]. However, it has been demonstrated in [20,21] that the generalized differential operator method allows to construct higher-order solitary solutions without the drawbacks of the direct balancing technique. The construction of higher-order solitary solutions to generalized Riccati equations remains a definite object of future research.

Acknowledgments

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References

[1] N.M.H. Alkoumi, P.J. Torres, Estimates on the number of limit cycles of a generalized Abel equation, Discrete Contin. Dyn. Syst., Ser. A 31 (2011) 25–34.
[2] A. Álvarez, J.L. Bravo, M. Fernández, Limit cycles of Abel equations of the first kind, J. Math. Anal. Appl. 423 (2015) 734–745.
[3] I. Aslan, V. Marinakis, Some remarks on Exp-function method and its applications, Commun. Theor. Phys. 56 (2011) 397–403.
[4] S. Bittanti, A.J. Laub, J.C. Willems, The Riccati Equation, Springer, Berlin, 1991.
[5] J.C. Butcher, Numerical Methods for Ordinary Differential Equations, Wiley, New York, 2008.
[6] J. de Lucas, M. Tobolski, S. Vilariño, Geometry of Riccati equations over normed division algebras, J. Math. Anal. Appl. 440 (2016) 394–414.

[7] N. Echi, Rational solutions of Riccati differential equation with coefficients rational, Appl. Math. Comput. 218 (2012) 10341–10366.
[8] A. Gasull, J. Torregrosa, X. Zhang, The number of polynomial solutions of polynomial Riccati equations, J. Differential Equations 261 (2016) 5071–5093.
[9] J. Giné, X. Santallusia, Abel differential equations admitting a certain first integral, Appl. Math. Comput. 370 (2010) 187–199.
[10] G. Griffiths, W.E. Schiesser, Traveling Wave Analysis of Partial Differential Equations, Academic Press, Cambridge, MA, 2010.
[11] N.A. Kudryashov, A note on the G'/G-expansion method, Appl. Math. Comput. 217 (2010) 1755–1758.
[12] N.A. Kudryashov, N.B. Loguinova, Be careful with Exp-function method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1891–1900.
[13] V.N. Laptinsky, I.I. Makovetsky, On the two-point boundary-value problem for the Riccati matrix differential equation, Cent. Eur. J. Math. 3 (2005) 143–154.
[14] S.C. Mancas, H.C. Rosu, Integrable dissipative nonlinear second order differential equations via factorizations and Abel equations, Phys. Lett. A 377 (2013) 1434–1438.
[15] R. Marcinkevicius, Z. Navickas, M. Ragulskis, T. Telksnys, Solitary solutions to a relativistic two-body problem, Astrophys. Space Sci. 361 (2016) 201.
[16] M.P. Markakis, Closed-form solutions of certain Abel equations of the first kind, Appl. Math. Lett. 22 (2009) 1401–1405.
[17] Z. Navickas, M. Ragulskis, How far can one go with the Exp-function method?, Appl. Math. Comput. 211 (2009) 522–530.
[18] Z. Navickas, L. Bikulciene, M. Ragulskis, Generalization of Exp-function and other standard function methods, Appl. Math. Comput. 216 (2010) 2380–2393.
[19] Z. Navickas, M. Ragulskis, L. Bikulciene, Be careful with the Exp-function method – additional remarks, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3874–3886.
[20] Z. Navickas, M. Ragulskis, L. Bikulciene, Special solutions of Huxley differential equation, Math. Model. Anal. 16 (2011) 248–259.
[21] Z. Navickas, L. Bikulciene, M. Rahula, M. Ragulskis, Algebraic operator method for the construction of solitary solutions to nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 1374–1389.
[22] Z. Navickas, M. Ragulskis, T. Telksnys, Existence of solitary solutions in a class of nonlinear differential equations with polynomial nonlinearity, Appl. Math. Comput. 283 (2016) 333–338.
[23] M. Nowakowski, H. Rosu, Newton's laws of motion in the form of a Riccati equation, Phys. Rev. E 65 (2002) 047602.
[24] F. Pakovich, Weak and strong composition conditions for the Abel differential equation, Bull. Sci. Math. 138 (2014) 993–998.
[25] A.A. Panov, The number of periodic solutions of polynomial differential equations, Math. Notes 64 (1998) 622–628.
[26] K. Parand, S.A. Hossayni, J.A. Rad, Operation matrix method based on Bernstein polynomials for the Riccati differential equation and Volterra population model, Appl. Math. Model. 40 (2016) 993–1011.
[27] A.D. Polyanin, V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, Chapman and Hall/CRC, 2003.
[28] W.T. Reid, Riccati Differential Equations, Academic Press, New York, 1972.
[29] J. Riccati, Animadversiones in aequationes differentiales secundi gradus, Actorum Eruditorum quae Lipsiae publicantur 8 (1724) 66–73.
[30] M.G. Sakar, Iterative reproducing kernel Hilbert spaces method for Riccati differential equations, J. Comput. Appl. Math. 309 (2017) 163–174.
[31] N.K. Vitanov, Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2050–2060.
[32] R.M. Yamaleev, Representation of solutions of n-order Riccati equation via generalized trigonometric functions, J. Math. Anal. Appl. 420 (2014) 334–347.