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Special Solutions of Huxley Differential Equation

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Abstract. The conditions when solutions of Huxley equation can be expressed in special form and the procedure of finding exact solutions are presented in this paper. Huxley equation is an evolution equation that describes the nerve propagation in biology. It is often useful to obtain a generalized solitary solution for fully understanding its physical meanings. It is shown that the solution produced by the Exp-function method may not hold for all initial conditions. It is proven that the analytical condition describing the existence of the produced solution in the space of initial conditions (or even in the space of the system's parameters) can not be derived by the Exp-function method because the question about the existence of that solution is omitted. The proposed operator method, on the contrary, brings the load of symbolic computations before the structure of the solution is identified. The method for the derivation of the solution is based on the concept of the rank of the Hankel matrix constructed from the sequence of coefficients representing formal solution in the series form. Moreover, the structure of the algebraic-analytic solution is generated automatically together with all conditions of the solution's existence. Computational experiments are used to illustrate the properties of derived analytical solutions.

Keywords: Nonlinear differential equation, Hankel matrix, algebraic-analytical solution.

AMS Subject Classification: 34A45; 34A25.

1 Introduction

Huxley equation is a core mathematical framework for modern biophysically based neural modelling. It is often useful to obtain a generalized solitary solution for fully understanding its physical meanings. The traditional approaches to this task are variational iteration method [1], the homotopy perturbation method [4, 16], Adomian‘s decomposition method [6], the tanh method [6], the Exp-function method [7, 19]. Approximate numerical tools describing Hodgkin-Huxley models help to identify peak currents in biological systems,

[17]. The Hodgkin-Huxley reaction-diffusion system [8] plays an important role in graph structures describing real neurons. Recent numerical approaches for the solution of this demanding nonlinear problem are described in [2] where finite-difference schemes and in [3] where parallel predictor-corrector schemes for parabolic problems on graphs are employed. It is often useful to obtain a generalized analytical solitary solution for fully understanding physical meanings of nonlinear processes taking place in Hodgkin-Huxley models. However, many methods for the construction of an analytical solution may sometimes fail or the solution procedure becomes complicated as the degree of nonlinearity increases. An analytical criterion determining if a solution of a differential equation can be expressed in an analytical form comprising exponential functions is developed in [12]. The employment of this criterion does not only give an answer to the above-stated question but gives the structure of the solution so that one does not have to guess what the form of the solution is. The load of symbolic calculations is brought before the structure of the solution is identified. This is in contrary to the Exp-function type methods where the structure of the solution is first guessed, and then symbolic calculations are exploited for the identification of parameters. Hodkin and Huxley presented [8] the results of electrophysiological experiments in which they investigated the flow of electric current through the surface membrane of the giant nerve fiber of a squid. Huxley equation is a nonlinear partial differential equation of second order of the form $u_t = u_{xx} + u\,(k-u)\,(u-1)$. This equation is an evolution equation that describes the nerve propagation in biology. From this equation molecular properties can be calculated. It also gives a description of the behaviour of the miosin heads. This equation has many fascinating phenomena such as bursting oscilation, interspike, bifurcation and chaos.

The Soliton model in neuroscience is a recently developed model that attempts to explain how signals are conducted within neurons. It proposes that signals travel along the cell's membrane in the form of certain kinds of sound (or density) pulses known as solitons. This model presents a direct challenge to the widely accepted Hodgkin-Huxley model [8] which proposes that signals travel as action potentials: voltage-gated ion channels in the membrane open and allow ions to rush into the cell, thereby leading to the opening of other nearby ion channels and thus propagating the signal in an essentially electrical manner.

Using the wave variable $\eta = \omega x + vt$ was obtained

$$-vu' + \omega^2 u'' + u\,(k-u)\,(u-1) = 0, \tag{1.1}$$

where $\omega, k, v \in R$.

The solitary solution of the Huxley equation, produced by the Exp-function method, does not satisfy the original differential equation for all initial conditions. We have used an alternative operator-based method to derive the solitary solution of the Huxleys equation and have identified the region (in the parameter plane) of the initial conditions, where this solution does exist.

2 Auxiliary Results

All the necessary notations, definitions and theorems associated with stating conditions, under which the power series solution of a differential equation can be reduced to a finite sum of standard functions, are presented below.

2.1 Structures of analytical solutions

Let two polynomials are defined as follows:

$$P_1(v,s) = \sum_{k,l\in Z_0} a_{kl} v^k s^l; \qquad P_2(v,s,t) = \sum_{k,l,r\in Z_0} b_{klr} v^k s^l t^r;$$

where a_{kl} and b_{klr} are fixed real (or complex) numbers; v, s and t are real (or complex) variables. Then it is possible to construct two ordinary differential equations with initial conditions:

$$y'_x = P_1(x,y), \quad y = y(x,v,s); \quad y(v,v,s) = s \tag{2.1}$$

and

$$\begin{aligned} &w''_{xx} = P_2(x,w,w'_x); \quad w = w(x,v,s,t); \\ &w(v,v,s,t) = s; \quad w'_x(x,v,s,t)\big|_{x=v} = t. \end{aligned} \tag{2.2}$$

Usual differentiation operations in respect of variables x, v, s and t are denoted by symbols D_x, D_v, D_s and D_t. Then it is possible to construct generalised differential operators D_y and D_w in respect of variables y and w [10]:

$$D_y := D_v + P_1(v,s) D_s, \quad D_w := D_v + tD_s + P_2(v,s,t) D_t.$$

These generalized differential operators satisfy all usual relationships of differential operators. Generalised differential operators D_y and D_w can be exploited to construct analytical solutions $y(x,v,s)$ and $w(x,v,s,t)$ of differentials equations (2.1) and (2.2) [10]:

$$y = \sum_{j=0}^{+\infty} \frac{(x-v)^j}{j!} D_y^j s, \quad w = \sum_{j=0}^{+\infty} \frac{(x-v)^j}{j!} D_w^j s,$$

which converge in some nonempty neighbourhood $|x-v| < \varepsilon$ in the complex plane. Furthermore, functions $y = y(x,v,s)$ and $w = w(x,v,s,t)$ can be extended into the whole complex plane with the exception of possible singular points. Here $D_y^j s$ and $D_w^j s$ are generalised differentiation [10] and, for instance,

$$D_y s = P_1(v,s), \quad D_y^2 s = (P_1(v,s))'_v + P_1(v,s))(P_1(v,s))'_s.$$

2.2 Structure of analytical – algebraic solutions

It is important for many engineering applications to obtain analytical – algebraic representations of solutions in the form:

$$y = \sum_{r=1}^{m} \mu_r f_r(\rho(x-v)),$$

where $m \in N$; $\mu_r \in C$; f_r and ρ are ordinary functions. We will define the H-rank and associated H-eigenvalues for the construction of special analytical – algebraic solutions. Let $(p_j;\ j \in Z_0)$ is a sequence of numbers or functions. Then the corresponding sequence of Hankel matrixes $(H_1, H_2, \ldots)$ reads:

$$H_1 := \begin{bmatrix} p_0 \end{bmatrix}; \quad H_2 := \begin{bmatrix} p_0 & p_1 \\ p_1 & p_2 \end{bmatrix}; \quad H_3 := \begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{bmatrix}; \quad \ldots$$

and the sequence of determinants of these matrixes is denoted as $(d_k;\ k \in N)$, where $d_k = \det H_k$.

Definition 1. A sequence $(p_j;\ j \in Z_0)$ has an H-rank equal to m if $d_m \neq 0$ but $d_{m+n} = 0$ for all $n \in N$. The following notion will be used throughout the manuscript:

$$Hr\,(p_j;\ j \in Z_0) = m. \tag{2.3}$$

Let (2.3) holds for a sequence $(p_j; j \in Z_0)$. Then it is possible to construct a characteristic H-equation [13]:

$$\det \begin{bmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ & & \cdots & \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{bmatrix} = 0. \tag{2.4}$$

Definition 2. Roots $\rho_1, \rho_2, \ldots, \rho_m$ of the characteristic H-equation (2.4) are H-eigenvalues of the sequence $(p_j; j \in Z_0)$

Theorem 1. *Let the H-rank of a sequence $(p_j;\ j \in Z_0)$ is m. Moreover, let all H-eigenvalues of that sequence are different: $\rho_k \neq \rho_l$ for $k \neq l$ $(k, l = 1, 2, \ldots, m)$. Then, following equalities hold true*:

$$p_j = \sum_{r=1}^{m} \mu_r \rho_r^j; \quad j = 0, 1, 2, \ldots. \tag{2.5}$$

The proof of Theorem 1 is given in [11].

It can be noted that coefficients $\mu_1, \mu_2, \ldots, \mu_m$ can be found by solving a linear system of algebraic equations which consists of m different equalities of (2.5) (H-eigenvalues $\rho_1, \rho_2, \ldots, \rho_m$ must be determined beforehand). The simplest system is produced when the first m equalities of (2.4) are selected (for $j = 0, 1, \ldots, m-1$). But the same results can be produced for $j = k_1, k_2, \ldots, k_m$; $0 \leq k_1 < k_2 < \cdots < k_m$. Moreover, this linear system of algebraic equations has a unique solution. The sequence defined by (2.4)is called an algebraic progression.

Let us assume that the following relationship holds for a sequence $(D_y^j s; j \in Z_0)$:

$$Hr\left(\frac{1}{j!} D_y^j s;\ j \in Z_0\right) = m.$$

Then, using Theorem 1 was obtained that

$$D_y^j s = j! \sum_{r=1}^{m} \mu_r \rho_r^j, \tag{2.6}$$

where $\mu_r{=}\mu_r(v,s)$; $\rho_r{=}\rho_r(v,s)$; $\rho_k{\neq}\rho_l$ for $k \neq l$. For instance, when $m = 1$

$$D_y^0 s = \mu_1(v,s), \quad D_y s = \mu_1(v,s)\rho(v,s) = P_1(v,s) = s\rho(v,s),$$

i.e.$\rho(v,s) = P_1(v,s)/s$ and $D_y^j s = j! s (P_1(v,s)/s)^j$.

Equation (2.6) can be exploited for the construction of the solution of differential equation (2.1) in the form of a power series:

$$y = \sum_{j=0}^{+\infty} \frac{(x-v)^j}{j!} D_y^j s = \sum_{j=0}^{+\infty} (x-v)^j \sum_{r=1}^{m} \mu_r \rho_r^j = \sum_{r=1}^{m} \mu_r \sum_{j=0}^{+\infty} (\rho_r (x-v))^j \tag{2.7}$$

which converges in the neighbourhood $|x-v| < \min_r |\rho_r|^{-1}$. But the power series (2.7) can be extended for all values of the variable x with the exception of such values of x where $\rho_r(x-v) = 1$. In other words, the function $y = y(x,v,s)$ takes the following form:

$$y = \sum_{r=1}^{m} \frac{\mu_r(v,s)}{1 - \rho_r(v,s) \cdot (x-v)}. \tag{2.8}$$

Equation (2.8) is the analytical-algebraic solution of differential equation (2.1).

Following the papers [5, 15] the differential equations with Cauchy conditions

$$y'_x = a\left(y^2 + \alpha y + \beta\right), \tag{2.9}$$

where $y = y(x,v,s)$ and $y = y(v,v,s) = s$, and

$$w''_{xx} + \hat{b} w'_x = \hat{a}\left(w^3 + \hat{\alpha} w^2 + \hat{\beta} w + \hat{\gamma}\right), \tag{2.10}$$

$w = w(x,v,s,t)$ and $w = w(v,v,s,t) = s$, $w'_x = w(x,v,s,t)|_{x=v} = t$ are called the Riccati and Huxley equations respectively. Here the parameters a, α, β, $\hat{b}$, $\hat{a}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, s, t are real (complex) fixed numbers, besides, $a, \hat{a} \neq 0$.

Using the algorithm of solution for differential equations and a change of variables [13], it was obtained that the solution of the Riccati differential equation has the following form:

$$y(x,v,s) = \frac{y_2(s-y_1)\exp(ay_1(x-v)) - y_1(s-y_2)\exp(ay_2(x-v))}{(s-y_1)\exp(ay_1(x-v)) - (s-y_2)\exp(ay_2(x-v))}. \tag{2.11}$$

But the Huxley differential equation has a solution expressed by (2.11) form only with special Cauchy conditions whitch will be discussed further.

2.3 Expanding and narrowing an ordinary differential equation

In paper [13] the algorithm of expansion and narrowing of differential equations is presented and from this algorithm the theorem follows:

Theorem 2. *Let two differential equations* (2.1)*and* (2.2) *be given. The relationship*

$$w\,(x, v, s, P_1(v, s)) = y\,(x, v, s)$$

holds true if and only if such identity is satisfied:

$$\frac{\partial P_1\,(v,s)}{\partial v} + P_1\,(v,s)\,\frac{\partial P_1\,(v,s)}{\partial s} := P_2\,(v, s, P_1(v,s))\,. \tag{2.12}$$

Proof of this theorem is given in [13].

Definition 3. The differential equation (2.2) is an expanded differential equation (2.1) and the differential equation (2.1) is a narrowed differential equation (2.2).

3 Expansion of Riccati Differential Equation to Huxley Differential Equation

3.1 Composition of expansion

For the Riccati differential equation (2.9) is given as

$$P_1\,(v,s) = a\big(s^2 + \alpha s + \beta\big) = a\,(s - y_1)\,(s - y_2)\,.$$

For the Huxley differential equation (2.10) is given as

$$P_2\,(v,s,t) = \hat{a}\big(s^3 + \hat{\alpha} s^2 + \hat{\beta} s + \hat{\gamma}\big) - \hat{b}t = \hat{a}\,(w - w_1)\,(w - w_2)\,(w - w_3) - \hat{b}t.$$

Here y_1, y_2 and w_1, w_2, w_3 are roots by variable s of polynomials $P_1\,(v,s)$ and $P_2\,(v,s,t)$ respectively.

Regarding expression (2.12) from Theorem 2 and choosing right sides of equations (2.9) ($y'_x = P_1(v,s)$) and (2.10) ($w''_{xx} = P_2(v,s,t)$) the expansion of Riccati differential equation to Huxley differential equation is obtained

$$\frac{\partial P_1}{\partial v} + P_1\frac{\partial P_1}{\partial s} = a\left(s^2 + \alpha s + \beta\right) a\,(2s + \alpha)\,, \tag{3.1}$$

$$P_2|_{t=P_1} = \hat{a}\big(s^3 + \hat{\alpha} s^2 + \hat{\beta} s + \hat{\gamma}\big) - \hat{b}a\big(s^2 + \alpha s + \beta\big). \tag{3.2}$$

Using (2.12) relationships and after performing usual algebraic operations the following expression is obtained from (3.1) and (3.2):

$$\begin{aligned} &2a\left(s^3 + \frac{3\alpha}{2}s^2 + \left(\frac{\alpha^2}{2} + \beta\right)s + \frac{\alpha\beta}{2}\right) \\ &\quad = \hat{a}\left(s^3 + \left(\hat{\alpha} - \frac{\hat{b}a}{\hat{a}}\right)s^2 + \left(\hat{\beta} - \frac{\hat{b}a\alpha}{\hat{a}}\right)s + \left(\hat{\gamma} - \frac{\hat{b}a\beta}{\hat{a}}\right)\right). \end{aligned} \tag{3.3}$$

From relationships (3.3) it follows that expansion of Riccati differential equation to Huxley differential equation is possible if and only if the equalities

$$\hat{a} = 2a^2, \quad \hat{\alpha} - \frac{\hat{b}a}{\hat{a}} = \frac{3\alpha}{2}, \quad \hat{\beta} - \frac{\hat{b}a\alpha}{\hat{a}} = \frac{\alpha^2}{2} + \beta, \quad \hat{\gamma} - \frac{\hat{b}a\beta}{\hat{a}} = \frac{\alpha\beta}{2} \tag{3.4}$$

are satisfied. For simplicity, if $\hat{b} := b$ then we obtain from (3.4) that

$$\hat{a} = 2a^2, \quad \hat{b} = b, \quad \hat{\alpha} = \frac{b}{2a} + \frac{3\alpha}{2}, \quad \hat{\beta} = \frac{\alpha^2}{2} + \beta + \frac{\alpha b}{2a}, \quad \hat{\gamma} = \frac{\alpha\beta}{2} + \frac{b\beta}{2a}. \tag{3.5}$$

If the equalities (3.5) are satisfied, then Huxley differential equation, obtained from Riccati differential equation has the following form

$$w''_{xx} + bw'_x = 2a^2 \left[w^3 + \left(\frac{3\alpha}{2} + \frac{b}{2a} \right) w^2 + \left(\frac{\alpha^2}{2} + \frac{\alpha\beta}{2a} + \beta \right) w + \left(\frac{\alpha\beta}{2} + \frac{b\beta}{2a} \right) \right]. \tag{3.6}$$

It follows from the Viète theorem that $\alpha = -(y_1 + y_2)$ and $\beta = y_1 y_2$, the relationships (3.6) is given by:

$$w''_{xx} + bw'_x = 2a^2 \left[w^3 + \left(\frac{b}{2a} - \frac{3(y_1+y_2)}{2} \right) w^2 + \left(\frac{(y_1+y_2)^2}{2} - \frac{y_1+y_2}{2a} + y_1y_2 \right) w + \left(\frac{by_1y_2}{2a} - \frac{(y_1+y_2)\,y_1y_2}{2} \right) \right]$$

and it is equivalent to the relationship

$$w''_{xx} + bw'_x = 2a^2 (w - y_1)(w - y_2) \left(w - \frac{1}{2}(y_1 + y_2 - b) \right).$$

Theorem 3. *Structural solution* (2.11) *satisfies differential equation* (2.2) *if and only if the relationship*

$$w^3 + \hat{\alpha}w^2 + \hat{\beta}w + \hat{\gamma} = (w - y_1)(w - y_2) \left(w - \frac{1}{2} \left(y_1 + y_2 - \frac{\hat{b}}{a} \right) \right) \tag{3.7}$$

holds true.

It must be noticed that expression (2.11) is only a particular solution of Huxley differential equation (2.2) which satisfies the relationship:

$$w(x, v, s, a(y - y_1)(y - y_2)) = y(x, v, s) \tag{3.8}$$

or has such Cauchy conditions

$$w(v, v, s, t) = s, \;\; w'_x(x, v, s, t)|_{x=0} = a(s - y_1)(s - y_2). \tag{3.9}$$

In case when $y_1 = y_2 := y_0$, from (2.1) we obtain the Riccati differential equation

$$y'_x = a\,(y - y_0)^2$$

and its common solution

$$y\,(x, v, s) = y_0 + \frac{s - y_0}{1 + a\,(y_0 - s)\,(x - v)}.$$

Then relationships (3.8) and (3.9) have the following form:

$$w\big(x, v, s, a\,(y - y_0)^2\big) = y\,(x, v, s) \tag{3.10}$$

$$w\,(v, v, s, t) = s,\;\; w'_x\,(x, v, s, t)\big|_{x=v} = a\,(y - y_0)^2 \tag{3.11}$$

and condition (3.7) has the form

$$w^3 + \hat{\lambda} w^2 + \hat{\beta} w + \hat{\gamma} = (w - y_0)^2 \left(w - y_0 + \hat{b}/(2a)\right).$$

3.2 Special solutions of Huxley differential equation: condition of existence

The expression (3.8) and analogous expression (3.10) are considered as special solutions of Huxley differential equation. From relationship (3.6) using Viète theorem such system of equalities

$$\begin{cases} y_1 + y_2 + 0.5\left(y_1 + y_2 - \hat{b}/a\right) = -\hat{\alpha}, \\ y_1 y_2 + 0.5 y_1\left(y_1 + y_2 - \hat{b}/a\right) + 0.5 y_2\left(y_1 + y_2 - \hat{b}/a\right) = \hat{\beta}, \\ 0.5 y_1 y_2\left(y_1 + y_2 - \hat{b}/a\right) = -\hat{\gamma}. \end{cases} \tag{3.12}$$

was obtained. From system (3.12) after elimination $y_1 + y_2$ and $y_1 y_2$ the equality

$$\big(\hat{b} - 2a\hat{\alpha}\big)\big(\big(\hat{b} + a\hat{\alpha}\big) + 9\hat{\beta}a^2\big)\big(\hat{b} + a\hat{\alpha}\big) = 27a^3\hat{\gamma} \tag{3.13}$$

was obtained. From (3.13), the following theorem follows.

Theorem 4. *Relationship* (3.7) *holds true if and only if the equality* (3.13) *is correct.*

So, the equality (3.13) together with system of equalities (3.12) allows to establish existence of special solution of the Huxley differential equation and if this solution exists, we can find values of parameters a, y_1 and y_2.

It must be noticed that if in relationship (2.2) the parameters $\hat{b} = 0$ and $b = 0$, then the Huxley differential equation reduces to the Maccari differential equation [18] with adequate special solutions and conditions of existence.

4 Computational Experiments

Let the Huxley differential equation

$$w''_{xx} + w'_x = 2\left(w^3 - 4w^2 + 5w - 2\right);$$
$$w = w\,(x, v, s, t)\,, \quad w\,(v, v, s, t) = s, \quad w'_x\,(x, v, s, t)|_{x=v} = t \qquad (4.1)$$

be given. Then $\hat{a} = 2, a = 1, \hat{b} = b = 1, \hat{\alpha} - 4, \hat{\beta} = 5, \hat{\gamma} = -2$. From system of equations (3.12), it is obtained that $y_1 + y_2 = 3$ and $y_1 y_2 = 2$, i.e. $y_1 = 1$, $y_2 = 2$, besides the relationship (3.13) is satisfied. Thus, the special solution of Huxley differential equation has a form:

$$w\,(x, v, s, (s-1)\,(s-2)) = \frac{2\,(s-1)\,e^{x-v} - (s-2)\,e^{2(x-v)}}{(s-1)\,e^{x-v} - (s-2)\,e^{2(x-v)}}, \qquad (4.2)$$

when Cauchy conditions are $w(v, v, s, (s-1)(s-2)) = s$ and $w'_x(x, v, s, (s-1)(s-2))|_{x=v} = (s-1)(s-2)$ and s is a real (complex) number.

The surface of graphical representation of the initial conditions at $s = 0$, $s = 3$ and $s = 5$, $t = y'\,(x)|_{x=v} = (s - y_1)\,(s - y_2)$ is plotted in Fig. 1. Here variables on axes Ox and Oy are y_1 and y_2 respectively.

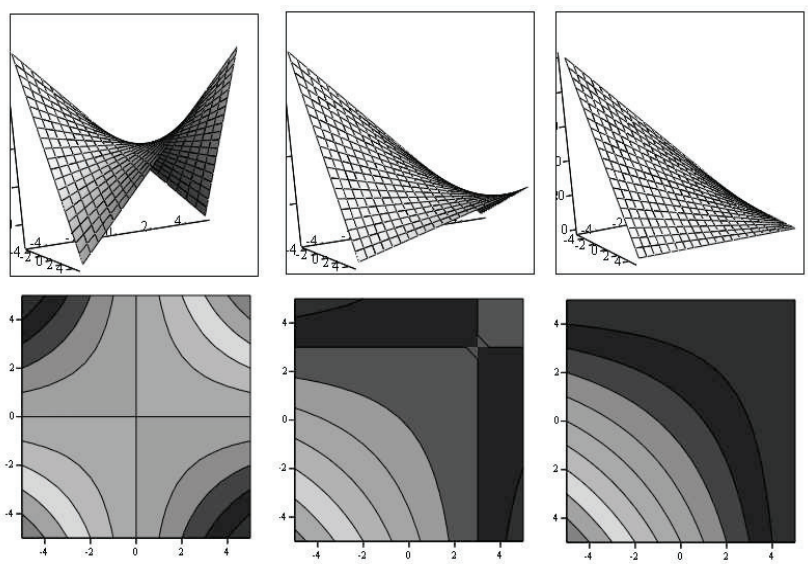

Figure 1. A graphical representation of the initial conditions at s=0, s=3 and s=5, $t = y'\,(x)|_{x=v} = (s - y_1)\,(s - y_2)$

The validity of the produced results by a computational experiment is performed. The initial problem (4.1) is solved using approximate computational constant step marching technique. Let us denote the approximate partial solution $\tilde{w}_k(0 + hk)$, $k = 0, 1, 2, \ldots$, where $\tilde{w}_k$ depends on both initial conditions, h is the step size and $\tilde{y}_0 = s$, where the analytical-algebraic partial solution

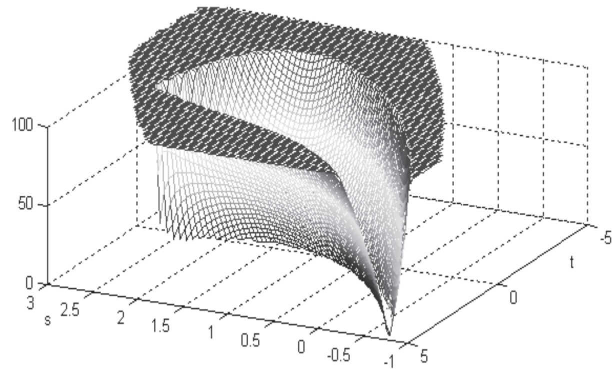

Figure 2. The distribution of errors (4.3) between the analytical solution and the computational solution in the parameter plane of initial conditions.

of (4.1) is defined on (4.2). But the constraint (4.2) was released and it was assumed that the solution (4.1) is valid in all space of initial conditions. The 100 steps from the predefined initial conditions were travelled and the differences between the approximate computational solution and the analytical "solution" defined by (4.1) was computed:

$$\varepsilon(s,t) = \sum_{k=1}^{100} |\varepsilon_k| = \sum_{k=1}^{100} \left| \tilde{w}_k(hk) - \frac{2\,(1-s)\exp(hk) - (2-s)\exp(2hk)}{(1-s)\exp(hk) - (2-s)\exp(2hk)} \right|. \quad (4.3)$$

The distribution of $\varepsilon(s,t)$ is illustrated in Fig. 2. Numerical values of $\varepsilon(s,t)$ higher than 100 are truncated to 100 in order to make the figure more comprehensive. It can be clearly seen that errors are almost equal to zero on the curve defined (4.2).

5 Concluding Remarks

The Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving high-dimensional nonlinear evolutions in mathematical physics. Unfortunately, there have been a number of cases when straightforward and formal application of the Exp-function method has produced irrelevant results. Seven typical errors done when using the Exp-function method are discussed and illustrated in detail in [9, 14].

The solution produced by the Exp-function method may not hold for all initial conditions. We argue that the analytical condition describing the existence of the produced solution in the space of initial conditions (or even in the space of the system's parameters) cannot be derived by the Exp-function method. The Exp-function method is based on two main steps (omitting transformations and variable changes leading to a nonlinear ordinary differential equation). In the first step we should define the structure of the algebraic-analytical solution. The second step consists in using symbolic computations for the determination of unknown parameters of the solution. The question about the existence of that solution is typically omitted. The operator method, on the contrary, brings the load of symbolic computations before the structure of the solution is identified. Moreover, the structure of the algebraic-analytic solution is generated automatically together with all conditions of the solution's existence.

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