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Research Article

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An extended Prony's interpolation scheme on an equispaced grid

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Abstract: An interpolation scheme on an equispaced grid based on the concept of the minimal order of the linear recurrent sequence is proposed in this paper. This interpolation scheme is exact when the number of nodes corresponds to the order of the linear recurrent function. It is shown that it is still possible to construct a nearest mimicking algebraic interpolant if the order of the linear recurrent function does not exist. The proposed interpolation technique can be considered as the extension of the Prony method and can be useful for describing noisy and defected signals.

Keywords: Interpolation, Prony method, The minimal order of linear recurrent sequence

MSC: 65D05, 65D10, 65D15

1 Introduction

It is well known that the n-point polynomial interpolants in equally spaced points to a function f on [-1, 1] do not necessarily converge as $n \to \infty$, even if f is analytic. Instead one may see wild oscillations near the endpoints, an effect known as the Runge phenomenon [9]. Moreover, the interpolation process becomes exponentially ill-conditioned, as shown first by Turetskii [13] and later independently by Schonhage [11]. This ill-conditioning means that even if the interpolants converge in theory, they will diverge in floating point arithmetic, at least for values of x near the endpoints, because of exponential amplification of rounding errors.

On the other hand, the polynomial interpolation in Chebyshev points is numerically stable since the associated Lebesgue constants are of size $O(\log(n))$ [3]. Is is shown in [4] that the Chebyshev interpolant can be evaluated in floating point arithmetic by Salzer's barycentric formula [10]. Moreover, Chebyshev interpolants are used in the Chebfun software where polynomials in degrees of tens of thousands are routinely used for practical computations [7, 12].

The main objective of this paper is to propose an extended Prony-type algebraic interpolation scheme on an equispaced grid. Our goal is to develop a strategy for finding a nearest algebraic interpolant to an analytic function. It will be demonstrated that an approach based on the nearest algebraic interpolant does not suppress the Runge phenomenon, but interpolation errors produced by this method are much lower if compared to the classic schemes

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on equispaced grids. Moreover, the proposed algebraic interpolation method can be effectively exploited for analytic interpolation of noisy and/or defected signals.

This paper is organized as follows. Linear recurring functions and linear recurring sequences are discussed in Section 2; computational examples of linear recurring sequences versus Prony decomposition are shown in Section 3; Extended Prony's functions and their properties are given in Section 4; algorithm of the extended Prony-type interpolation is presented in Section 5; the interpolation of the real time series is investigated in Section 6; concluding remarks are given in Section 7.

2 Preliminaries

The definition of the linear recurring function, the linear recurring sequence and their properties will be presented in this section.

2.1 A short overview of the Prony method

Fourier expansions

$$F(x) = \sum_{n=-\infty}^{+\infty} c_n \exp\left(i\frac{2\pi n}{T}x\right)$$

are successfully exploited for the approximation of a function $f(x)$ in a variety of theoretical and practical applications. And even though higher terms of the Fourier series are usually infinitesimal, that may cause substantial complications. In contrast, Prony's method computes an approximation to function $f(x)$ by using only a finite number of damped complex exponentials [14]:

$$F^*(x) = \sum_{n=1}^{m} c_n \exp(\lambda_n x)$$

where $m \in \mathbb{N}$ and $c_n, \lambda_n \in \mathbb{C}$. Prony-type methods are successfully exploited for effective and accurate approximation of different functions and signals [15–18] (though the determination of m remains problematic in general).

As mentioned previously, the main objective of this paper is to use the extended Prony-type scheme to interpolate a function on an equispaced grid. The extended Prony scheme comprises complex exponentials and polynomials [1, 2, 6]:

$$G(x) = \sum_{n=1}^{m} Q_n(x) \exp(\lambda_n x)$$

where $m \in \mathbb{N}$; $\lambda_n \in \mathbb{C}$ and $Q_n(x)$ are polynomials in x with complex coefficients and non-negative integer powers of x.

2.2 The minimal order of linear recurring sequence

Let us consider a sequence:

$$p_0, p_1, p_2, \ldots := (p_j; j \in Z_0)$$

where elements p_j can be real or complex numbers. Then, a sequence of Hankel matrices reads:

$$H_n := (p_{i+j-2})_{1 \le i,j \le n} = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ & & \cdots & \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix}; \; n = 1, 2, \ldots .$$

The Hankel transform (the sequence of determinants of Hankel matrices) $(d_n; n \in N)$ reads:

$$d_n := \det H_n.$$

Definition 2.1. *The minimal order of the recurring sequence $(p_j; j \in Z_0)$ is $m \in Z_0$; $m < +\infty$*

$$rank(p_j; j \in Z_0) = m$$

if the sequence of determinants of Hankel matrices has the following structure:

$$(d_1, d_2, ..., d_m, 0, 0, ...) \tag{1}$$

where $d_m \neq 0$ and $d_{m+1} = d_{m+2} = ... = 0$ [19, 20].

Example 2.2. *Let $p_j := j^2$; $j \in Z_0$. Then, $rank(j^2; j \in Z_0) = 3$ because the sequence of determinants of Hankel matrices reads* $(0, -1, -8, 0, 0, \ldots)$.

Definition 2.3. *Let $rank\left(p_j; j \in Z_0\right) = m$. Then the characteristic polynomial for the sequence $(p_j; j \in Z_0)$ is defined as [19, 20]:*

$$\hat{d}_m := \det \hat{H}_m := \begin{vmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{vmatrix} = 0. \tag{2}$$

The expansion of the determinant in (2) yields an m-th order algebraic equation for the determination of roots of the characteristic polynomial:

$$A_m \rho^m + A_{m-1}\rho^{m-1} + ... + A_1\rho + A_0 = 0; \tag{3}$$

where $A_m \neq 0$ because $d_m \neq 0$.

Theorem 2.4. *Let the minimal order of the sequence $\left(p_j; j \in Z_0\right)$ be m and the multiplicity indexes of roots $\rho_1, \rho_2, ..., \rho_l$ of the characteristic polynomial* (3) *are $m_1, m_2, ..., m_l$ accordingly; $\sum_{r=1}^{l} m_r = m$. Then the following equality holds true [19, 20]:*

$$p_j = \sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \mu_{rk} \binom{j}{k} \rho_r^{j-k}; \tag{4}$$

where $\mu_{rk}, \rho_r \in \mathbb{C}$; $\mu_{rm_r-1} \neq 0$ [5].
Note that $\mu_{rk} \binom{j}{k} \rho_r^{j-k} = 0$ if $\binom{j}{k} = 0$ what is true when $0 \leq j < k$. Moreover, $0^0 = 1$; $0^1 = 0^2 = ... = 0$.

The opposite statement holds also. If (4) *holds true, then*

$$rank(p_j; j \in Z_0) = m_1 + m_2 + ... + m_l.$$

Rigorous proof of this theorem is given in [5].

Definition 2.5. *A sequence $(p_j; j \in Z_0)$ is a linear recurring sequence (LRS) if elements of that sequence can be expressed in the form of* (4).

Corollary 2.6. (4) *can be rewritten in the following form:*

$$\sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \mu_{rk} \binom{j}{k} \rho_r^{j-k} = \sum_{r=1}^{l} \left(\sum_{k=0}^{m_r-1} \hat{\mu}_{rk} j^k \right) \rho_r^j; \tag{5}$$

where $\rho_1, \rho_2, ..., \rho_l \neq 0$ and $\mu_{rm_r-1} \neq 0$.

For example,

$$\sum_{k=0}^{2} \mu_{rk} \binom{j}{k} \rho_r^{j-k} = \mu_{r0} \binom{j}{0} \rho_r^j + \mu_{r1} \binom{j}{1} \rho_r^{j-1} + \mu_{r2} \binom{j}{2} \rho_r^{j-2}$$
$$= \mu_{r0}\rho_r^j + \frac{\mu_{r1}}{\rho_r} j\rho_r^j + \left(\frac{\mu_{r2}}{2\rho_r^2} j^2 - \frac{\mu_{r2}}{2\rho_r^2} j\right) \rho_r^j = \left(\sum_{k=0}^{2} \hat{\mu}_{rk} j^k\right) \rho_r^j$$

where $\hat{\mu}_{r0} = \mu_{r0}$; $\hat{\mu}_{r1} = \frac{\mu_{r1}}{\rho_r} - \frac{\mu_{r2}}{2\rho_r^2}$; $\hat{\mu}_{r2} = \frac{\mu_{r2}}{2\rho_r^2}$.

Thus, the sequence $q_j = \sum_{r=1}^{l} \left(\sum_{k=0}^{m_r-1} \hat{\mu}_{rk} \cdot j^k\right) \rho_r^j$; $j = 0, 1, 2, \ldots$ is an LRS and its minimal order of LRS is equal to $rank(q_j; j \in Z_0) = m_1 + m_2 + ... + m_l$ because $\hat{\mu}_{rm_r-1} = \mu_{rm_r-1} \neq 0$.

Remark 2.7. (4) *simplifies if all roots of the characteristic polynomial are distinct:*

$$p_j = \sum_{r=1}^{m} \mu_r \rho_r^j.$$

Note that coefficients μ_{rk} (or just μ_r) are determined in order to fit the initial conditions of the recurrence (roots $\rho_1, \rho_2, ..., \rho_l$ are defined by (2)*):*

$$\sum_{r=1}^{l} \sum_{k=0}^{m_r-1} \binom{j}{k} \rho_r^{j-k} \mu_{rk} = p_j; j = 0, 1, ..., m-1. \tag{6}$$

This system of linear equations has a unique solution [5].

Remark 2.8. *Let $rank(p_j; j \in \mathbb{Z}_0) = m$ and the first 2m elements of that series are known. Then it is possible to use* (3), (6) *and* (4) *to calculate all elements of that sequence.*

Corollary 2.9. *Let the sequence $(p_j; j \in \mathbb{Z}_0)$ be an LRS and its minimal order be equal to m. Then the following linear recurrence relation holds true [5]:*

$$B_0 p_j + B_1 p_{j+1} + ... + B_{m-1} p_{j+m-1} = p_{j+m}; j = 0, 1, ... \tag{7}$$

where constants $B_0, B_1, ..., B_{m-1} \in \mathbb{C}$ exist and do not depend on j.

Example 2.10. *Let $p_j := j^2$, $j \in \mathbb{Z}_0$. Then, $B_0 = 1$, $B_1 = -3$, and $B_2 = 3$ because*

$$j^2 - 3(j+1)^2 + 3(j+2)^2 = (j+3)^2.$$

On the other hand (according to (2)*), the characteristic polynomial*

$$-8\rho^3 + 24\rho^2 - 24\rho + 8 = -8(\rho - 1)^3 = 0$$

yields one root $\rho_1 = 1$; its multiplicity index is $m_1 = 3$. Therefore, according to (4)*,*

$$p_j = \sum_{k=0}^{2} \mu_{1k} \binom{j}{k} 1^{j-k} = \mu_{10}\binom{j}{0} + \mu_{11}\binom{j}{1} + \mu_{12}\binom{j}{2}.$$

Coefficients $\mu_{10}, \mu_{11}, \mu_{12}$ are determined in order to fit the recurrence: $\mu_{10} = 0$; $\mu_{11} = 1$; $\mu_{12} = 2$. Thus, finally,

$$p_j = j + 2 \cdot \frac{j(j-1)}{2} = j^2 \quad for \quad j \geq 2 \ (p_0 = 0; p_1 = 1).$$

Remark 2.11. *The determination of the minimal order of sequence $\{p_j\}$ by using Definition 2.1 has a cost of*

$$\sum_{j=1}^{m} O(j^3) = O(m^4)$$

flops. Construction of characteristic polynomial (3) *needs $O(m^4)$ flops and $O(m^2)$ flops are required to find the roots of this polynomial. System of linear equations* (6) *can be solved by using the Gaussian elimination method, the process has a cost of $2m^3/3$ flops [22, 23].*

3 LRS versus Prony decomposition - two computational examples

Let us consider a sequence $p_j, j = 0, ..., N$ where $\tilde{s}, N \in \mathbb{N}$, $3 \le \tilde{s} \le N$, $\tilde{s}$ is an upper bound of the number of exponentials and the bounds $\epsilon_{0,1}$ are positive. An LRS of a given sequence could be found using the Prony interpolation method [21]. The main steps of this algorithm read:

1. Determine the smallest singular value of the rectangular Hankel matrix ($H := (p_{o+g})_{o=1,g=0}^{N-\tilde{s},\tilde{s}}$ and use singular value decomposition to find the related right singular vector $\mathbf{u} = (u_l)_{l=0}^{\tilde{s}}$.
2. Compute all zeros of polynomial $\sum_{l=0}^{\tilde{s}} u_l z^l$ and determine all zeros $\tilde{z}_t, t = 1, ..., \tilde{M}$ for which $||\tilde{z}_t| - 1| \le \epsilon_1$. Note that $\tilde{s} \ge \tilde{M}$.
3. For $\tilde{\omega}_t := \tilde{z}_t/|\tilde{z}_t|$ $(t = 1, ..., \tilde{M})$ compute $\tilde{c}_t \in \mathbb{C}$ $(t = 1, ..., \tilde{M})$ as least squares solution of the overdetermined linear Vandermonde-type system:
$$\sum_{t=1}^{\tilde{M}} \tilde{c}_t \tilde{\omega}_t^j = p_j,\ j = 0, ..., N.$$
4. Delete all the $\tilde{\omega}_l$ $(l \in 1, ..., \tilde{M})$ with $|\tilde{c}_l| \le \epsilon_0$ and denote the remaining entries by $\tilde{\omega}_t$ $(t = 1, ..., M)$ with $M \le \tilde{M}$.
5. Repeat step 3 and compute $\tilde{c}_t \in \mathbb{C}, (t = 1, ..., M)$ as least squares solution of the overdetermined linear Vandermonde-type system
$$\sum_{t=1}^{M} \tilde{c}_t \tilde{\omega}_t^j = p_j,\ j = 0, ..., N$$
in accordance to the new set $\tilde{\omega}_t, t = 1, ..., M$ again.

Note that if $\tilde{z}_t$ are multiple zeros of order n_k then coefficients $\tilde{c}_{t,r} \in \mathbb{C}$ $(t = 1, ..., \hat{M}, r = 0, ..., n_k)$ are obtained as least squares solution of the overdetermined linear Vandermonde-type system:

$$\sum_{t=1}^{\hat{M}} \left(\sum_{r=0}^{n_k} \tilde{c}_{t,r} j^r \right) \tilde{\omega}_t^j = p_j,\ j = 0, ..., N,\quad \hat{M} \le \tilde{s}.$$

Example 3.1. *Let us consider a sequence $p_j := j, j = 0, ..., N-1, N = 21$. Let us find a LRS of a given sequence using two alternative methods: a) the concept of the minimal order of LRS; b) the Prony interpolation method [21].*

a) The minimal order of LRS of a given sequence is 2 because the sequence of determinants of Hankel matrices reads $(0, -1, 0, 0, ...)$. Then the characteristic polynomial reads:

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & \rho & \rho^2 \end{vmatrix} = -\rho^2 + 2\rho - 1 = 0.$$

The roots are: $\rho_1, \rho_2 = 1$. Now, the linear algebraic system of equations (6) yields:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ ... & ... \\ 1 & 20 \end{bmatrix} \begin{bmatrix} \mu_{10} \\ \mu_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ ... \\ 20 \end{bmatrix}.$$

Thus $\mu_{10} = 0$; $\mu_{11} = 1$.

Then the LRS of the given sequence reads:

$$\hat{p}_j^{(H)} = \mu_{10} \binom{j}{0} \rho_1^j + \mu_{11} \binom{j}{1} \rho_1^j = \binom{j}{1} 1_1^j = j, j = 0, ..., 20.$$

b) Let $\tilde{s} = [N/2-1] = 9$, $\epsilon_0 = 10^{-4}$, $\epsilon_1 = 10^{-4}$. According to the Prony interpolation algorithm [21] the rectangular Hankel matrix reads:

$$H := (p_{o+g})_{o=1,g=0}^{N-\tilde{s},\tilde{s}} = \begin{bmatrix} 0 & 1 & 2 & 3 & ... & 9 \\ 1 & 2 & 3 & 4 & ... & 10 \\ 3 & 4 & 5 & 6 & ... & 11 \\ ... & ... & ... & ... & ... & ... \\ 11 & 12 & 13 & 14 & ... & 20 \end{bmatrix}.$$

Now, the sequence of the singular values of the Hankel matrix reads: $(119.67, 9.94, 0, 0, 0, 0, 0, 0, 0, 0)$. *It can be noted that the smallest singular value is a tenth value. The related singular vector is given by* $u = (u_l)_{l=0}^{9} = (-0.2179 \ \ -0.2889 \ \ 0.4505 \ \ 0.6683 \ \ -0.3815 \ \ -0.0538 \ \ -0.1445 \ \ -0.1360 \ \ -0.0629 \ \ 0.1667)$.

Now the polynomial $\sum_{l=0}^{9} u_l z^l$ can be constructed, it has two zeros $\tilde{z}_1 = \tilde{z}_2 = 1$ satifying the property $||\tilde{z}_t|-1| \le \epsilon_1,\ t = 1,...,\tilde{s}$. Note that in this case $\tilde{z}_{1,2}$ are multiple zeros of order 2. Thus the overdetermined linear Vandermonde-type system $\sum_{t=1}^{\tilde{s}} \left(\sum_{r=0}^{2} \tilde{c}_{t,r} j^r\right) \tilde{\omega}_t^j = p_j,\ j = 0,...,N-1$ yields:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 3 & 9 & 1 & 3 & 9 \\ ... & ... & ... & ... & ... & ... \\ 1 & 20 & 400 & 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} \tilde{c}_{10} \\ \tilde{c}_{11} \\ \tilde{c}_{12} \\ \tilde{c}_{20} \\ \tilde{c}_{21} \\ \tilde{c}_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ ... \\ 20 \end{bmatrix}.$$

Thus $\tilde{c}_{11} = \tilde{c}_{21} = 0.5, \tilde{c}_{10} = \tilde{c}_{12} = \tilde{c}_{20} = \tilde{c}_{22} = 0$. Then the LRS of the given sequence reads:

$$\hat{p}_j^{(P)} = 0.5j + 0.5j = j,\ j = 0,...,20.$$

Definition 3.2. *An extended Prony's function (EPF) $f(x)$ is expressible in the following form:*

$$f(x) = \sum_{r=0}^{n} Q_r(x) \exp(\lambda_r x); \tag{8}$$

where $Q_r(x) = \sum_{k_r=0}^{m_r-1} a_{rk_r} x^{k_r}$; $m_r \ge 1$; $a_{rk_r} \in \mathbb{C}$; $a_{r,(m_r-1)} \ne 0$; $r = 0, 1, \dots, n$ and $x, f(x) \in R$. It is important to note that n is finite in (8).

Example 3.3. *Let us consider a function $\tilde{p}_j = j,\ j = 0, 1, 2, ..., 10, 11+e, 12, 13, ..., N-1,\ N = 101,\ e = -0.1$. Let us find a EPF of a given function using methods: a) the extended Prony-type interpolation method; b) the Prony interpolation method [21].*

a) Let the minimal order of the EPF be 14. *Then the characteristic polynomial is defined by $-\rho^{14} + 2\rho^{13} - \rho^{12} = 0$; the roots of this polynomial are $\rho_1 = \rho_2 = ... = \rho_{12} = 0$, $\rho_{13} = \rho_{14} = 1$. Now, the linear algebraic system of equations* (6) *yields:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & ... & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & ... & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & ... & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 4 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 99 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 100 \end{bmatrix} \begin{bmatrix} \mu_{10} \\ \mu_{11} \\ \mu_{12} \\ ... \\ \mu_{111} \\ \mu_{20} \\ \mu_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ ... \\ 99 \\ 100 \end{bmatrix}.$$

Thus $\mu_{111} = -0.1, \mu_{21} = 1$.

Then the EPF of the given sequence reads:

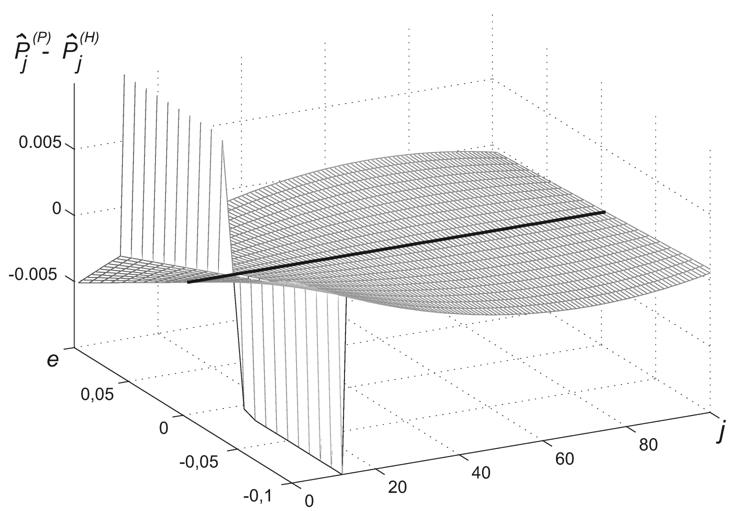
$$\hat{P}_j^{(H)} = \mu_{111} \binom{j}{11} \rho_1^{j-11} + \mu_{21} \binom{j}{1} \rho_2^j = j - 0.1 \binom{j}{11} 0^{j-11},\ j = 0,...,100.$$

b) Let $\tilde{s} = [N/2-1] = 49$, $\epsilon_0 = 10^{-4}$, $\epsilon_1 = 10^{-4}$. *The sequence of the singular values of the Hankel matrix reads:* $(264.7,, 196.8, 0.1, ..., 5 \cdot 10^{-17})$. *The related right singular vector by singular value decomposition is* $u = (u_l)_{l=0}^{49} = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0.0410\ \ -0.0999\ \ -0.1842\ 0.1331\ 0.1033\ \ -0.0098\ \ -0.0350\ \ -0.2179\ 0.2590\ 0.1325\ 0.2229$ *-0.3711* $-0.0094\ \ -0.1837\ 0.0509\ 0.2805\ \ -0.3368\ \ -0.1034\ \ -0.2375\ 0.3521\ 0.2704\ -0.0646\ 0.1245\ 0.0051\ -0.1061\ 0.0637\ 0.0680\ 0.0237\ 0.0828\ 0.0404\ -0.0392\ -0.1158\ 0.0486\ \ -0.1109\ \ -0.13080.1170\ \ -0.0564\ \ -0.0070)$. *Thus characteristic polynomial* $\sum_{l=0}^{49} u_l z^l$ *has two multiple zeros (with property* $||\tilde{z}_t|-1| \le \epsilon_1$*):* $\tilde{z}_1 = \tilde{z}_2 = 1$*; Then the constructed overdetermined linear Vandermonde-type system yields:* $\tilde{c}_{11} = \tilde{c}_{21} = 0.5, \tilde{c}_{10} = \tilde{c}_{20} = -0.0026, \tilde{c}_{12} = \tilde{c}_{22} = 0$ *and the EPF of the given sequence reads:*

$$\hat{P}_j^{(P)} = -0.0026 + 0.5j - 0.0026 - 0.5j = j - 0.0052,\ j = 0, ..., N-1.$$

Now let us consider functions $\bar{p}_j = j,\ j = 0, 1, 2, ..., 10, 11+e, 12, 13, ..., N-1,\ N = 101$ *where* $e \in [-0.1; 0.1]$. *Let us find a EPF of the each given function using the extended Prony-type and Prony interpolation methods. Differences between the computed EPF functions are shown in Figure 1. The thick solid line in Figure 1 stands for* $e = 0$*;* $\hat{P}_j^{(P)}$ *and* $\hat{P}_j^{(H)}$ *coincide for all* j *then. But even a slight perturbation* e *results into computational errors in* $\hat{P}_j^{(P)}$*. Moreover, these errors are not concentrated only around the 12-th element of the sequence - they are distributed throughout the whole domain (Figure 1). LRS theory (especially when the roots of the characteristic polynomial are multiple) enables the formal manipulation with such algebraic expressions as* 0^0 *– which becomes a very important issue in practical problems of interpolation.*

Fig. 1. The difference between $\hat{P}_j^{(P)} - \hat{P}_j^{(H)}$ at different values of the perturbation parameter e. The thick solid line stands for $e = 0$.



4 Extended Prony's functions and their properties

Theorem 4.1. *Let* $f(x)$ *be an EPF. Then the sequence*

$$y_j := f(x_0 + jh);\ j \in 0, 1, 2, \dots; \tag{9}$$

is a LRS ($x_0, h \in R$ *are fixed parameters).*

Proof. Let the function $y = f(x)$ be an EPF. Then, the following equalities hold for all $x_0, h \in R$:

$$y_j := f(x_0 + jh) = \sum_{r=1}^{l} \left(\sum_{k=0}^{m_r - 1} a_{rk} (x_0 + jh)^k \exp\left(\lambda_r (x_0 + jh)\right) \right)$$

$$= \sum_{r=1}^{l} \left(\sum_{k=0}^{m_r-1} b_{rk} \cdot j^k \right) (\exp(\lambda_r h))^j \,, \tag{10}$$

where coefficients b_{rk} can be expressed in terms of coefficients $a_{r0}, a_{r1}, ..., a_{rm_r-1}$. It can be noted that the index k, parameters x_0 and h do not depend on j. The introduction of the symbol

$$\rho_r = \exp(\lambda_r h) \tag{11}$$

reduces (10) into the LRS:

$$y_j = \sum_{r=1}^{l} \left(\sum_{k=0}^{m_r-1} b_{rk} \cdot j^k \right) \rho_r^j; \; j = 0, 1, 2, \ldots. \tag{12}$$

□

Theorem 4.2. *Let $(y_j; j \in Z_0)$ be an LRS. Then the following inequalities hold true:*

$$0 \le rank(y_j;\; j \in Z_0) \le m_1 + m_2 + ... + m_r. \tag{13}$$

Proof. (5) can be used to express terms in (12) (coefficients $b_{rk} \neq 0$). Combining like terms (some roots ρ_r may coincide) yields the estimate in (13). □

Example 4.3. *Let $f(x) = \cos x = \frac{1}{2}(\exp(ix) + \exp(-ix));\;\; x \in [0; 30\pi]$. Let us compute the minimal order of LRS when $h_1 = \frac{\pi}{2};\;\; h_2 = 2\pi;\; h_3 = \pi$.*
$y_{1,k} = f(x_0 + kh_1) = (1, 0, -1, 0, 1, 0, -1, 0, 1, ...)$;
$y_{2,k} = f(x_0 + kh_2) = (1, 1, 1, 1, 1, ...)$ when $x_0 = 0$.
But $y_{3,k} = f(x_0 + kh_3) = (0, 0, 0, 0, 0, ...)$ when $x_0 = \frac{\pi}{2}$.
It is clear that $rank(y_{1,k}) = 2;\;\; rank(y_{2,k}) = 1;\; rank(y_{3,k}) = 0$. It can be seen that the minimal order of LRS depends on the step size h and x_0. Also, it can be observed that all three sequences comprise exact values of the function $f(x)$.

Definition 4.4. *Let $f(x)$ be a EPF. Then, the minimal order of $f(x)$ is denoted as $rank(f(x))$ and is defined as follows:*

$$rank(f(x)) := \max_{x_0, h} rank(f(x_0 + kh);\; k \in Z_0);$$

where $x_0, h \in R;\; h > 0$.

Example 4.5. $rank(\cos x) = 2$.

Definition 4.6. *A sequence $f(x_0 + kh;\; k \in Z_0)$ is a representative sequence if the following equality holds true for fixed x_0 and h:*

$$rank(f(x)) = rank(f(x_0 + kh;\; k \in Z_0)).$$

Theorem 4.7. *Let $(y_j; j \in Z_0)$ be a representative LRS. Then, indexes $\lambda_1, \lambda_2, ..., \lambda_l$ read:*

$$\lambda_r = \frac{1}{h}(\ln|\rho_r| + i(\arg\rho_r + 2\pi k_r));\; k_r = 0, \pm 1, \pm 2, ...;\; r = \overline{1, l}. \tag{14}$$

Proof. (11) and the definition of the complex logarithm yield:

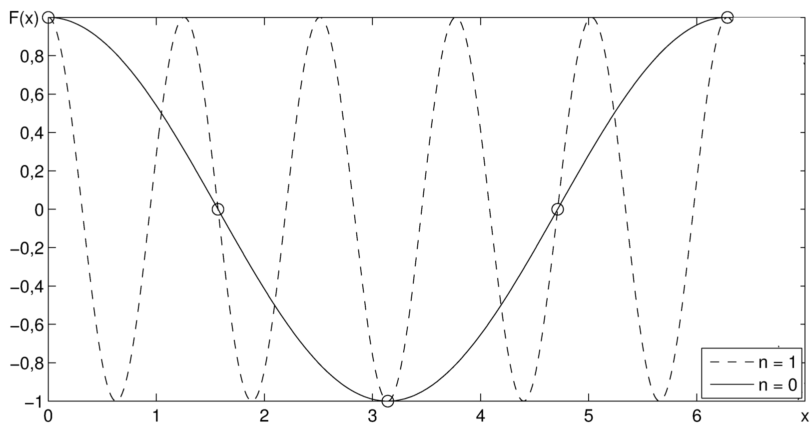
$$\lambda_r h := Ln\rho_r = \ln|\rho_r| + i(\arg\rho_r + 2\pi k_r); k_r = 0, \pm 1, \pm 2, ...; r = \overline{1, l}.$$

It is important to select such coefficients k_r that different ρ_r would correspond to different λ_r for all r. That ensures the equality of the EPF. □

Example 4.8. *Let $p_j := \frac{1}{2}(i^j + (-i)^j) = \cos\frac{\pi j}{2}$. We will find real EPF $y = f(x);\; x; f(x) \in R$ satisfying the constrains $f\left(\frac{\pi j}{2}\right) = p_j;\; j = 0, 1, 2, ...$ ($x_0 = 0$ and $h = \frac{\pi}{2}$). It can be noted that $rank(p_j; j \in Z_0) = 2$. Let us assume that this sequence is a representative LRS. This assumption yields the requirement that $rank f(x) = 2$. Then,*

$\rho_1 = i$ and $\rho_2 = -i$. Moreover, $\frac{\pi}{2}\lambda_1 = Ln\, i = i\left(\frac{\pi}{2} + 2\pi k\right)$; $\lambda_1 = i\,(1+4k)$. Analogously, $\lambda_2 = i\,(-1+4l)$. It can be noted that $\lambda_1 \neq \lambda_2$; $k, l = 0, \pm 1, \pm 2, \ldots$. Then, the algebraic interpolant reads: $y = f(x;k,l) = \frac{1}{2}\left(\exp\left(i\,(1+4k)\,x\right) + \exp\left(i\,(-1+4l)\,x\right)\right)$; $k,l = 0, \pm 1, \pm 2, \ldots$. It can be noted that $rank f(x;k,l) = 2$; $f\left(\frac{\pi j}{2};k,l\right) = \cos\frac{\pi j}{2}$. But $f(x,k,l)$ will be a EPF only when $l = -k = n$. Therefore, $f(x;n,-n) = \frac{1}{2}\left(\exp\left(i\,(1+4n)\,x\right) + \exp\left(-i\,(-1+4n)\,x\right)\right) = \cos\left((4n+1)\,x\right)$; $n = 0, 1, 2, \ldots$. The graphs of $f(x,n,-n)$ at $n = 0, 1$ are shown in Figure 2.

Fig. 2. Graphs of $f(x,n,-n)$ at $n = 0$ (the solid line) and $n = 1$ (the dashed line); circles denote nodes of the equispaced grid.



It can be noted that the selection of the parameter h may be a non-trivial task in the general case. Therefore, it is important to define the concept of the sufficiently small step h.

Definition 4.9. *The step h is sufficiently small if there exist such $h_0 > 0$ that the following relationship holds true for all $0 < h < h_0$:*

$$\lambda_r = \frac{1}{h}\left(\ln|\rho_r| + i \arg \rho_r\right); \tag{15}$$

where $0 \leq |\arg \rho_r| < \pi$ and $r = \overline{1,l}$.

Lemma 4.10. *If $f(x)$ is EPF defined by (8) then such $h_0 > 0$ exists that (15) holds true.*

Proof. (14) yields: $a_r h = |\ln \rho_r|$; $b_r h = \arg \rho_r + 2\pi k_r(h)$ where $\lambda_r = a_r + i b_r$ and $0 < |b_r| < +\infty$; $a \in R$. Thus, $b_r h = \arg \rho_r$ while $0 < |b_r|\, h < \pi$. Therefore, $k_r(h) = 0$ and $0 < h < \frac{\pi}{|b_r|}$.

The selection of $h_0 := \min\limits_r \frac{\pi}{|b_r|}$ finalizes the proof. □

It can be noted that h_0 would be unbounded if $\max\limits_r b_r = 0$.

Lemma 4.11. *Let (15) holds true for $h_0 > 0$. Then*

$$k_r = \frac{1}{2\pi h_0}\left(h \arg \rho_r(h_0) - h_0 \arg \rho_r(h)\right); \; r = \overline{1,l}.$$

Proof. Let

$$\rho_r := \rho_r(h) = \exp\left((a_r + i b_r)h\right), \tag{16}$$

i.e. $\rho_r := \rho_r(h)$ is a function of h; $0 < h < +\infty$. Then,

$$b_r h = \arg \rho_r(h) + 2\pi k_r(h). \tag{17}$$

It follows from the definition of h_0 that

$b_r = \frac{\arg \rho_r(h_0)}{h_0}$ or $b_r h = \arg \rho_r(h_0) \cdot \frac{h}{h_0}$. Then, (17) yields

$$\frac{h}{h_0} \cdot \arg \rho_r(h_0) = \arg \rho_r(h_0) + 2\pi k_r(h). \qquad \square$$

Lemma 4.12. *The following equality holds for all* $0 < h < +\infty$*:*

$$h \left| \ln \rho_r(h_0) \right| = h_0 \ln \left| \rho_r(h) \right|.$$

Proof. (16) yields $a_r = \frac{\ln|\rho_r(h)|}{h}$. Thus, $a_r = \frac{\ln|\rho_r(h_0)|}{h_0}$. $\square$

Example 4.13. *It can be noted that the step* $h_0 = \pi$ *is a sufficiently small step for* $f(x) = \cos x$*. Then parameters* k *and* l *must be set to 0.*

5 The extended Prony-type interpolation method

Let us assume that a function $f(x)$ is not necessarily EPF. Then the reconstruction of the closest EPF to the function $f(x)$ in the interval $a \le x \le b$ would be an important practical problem which is discussed in details in this section.

First of all the step h of the regular grid in the interval $[a; b]$ should be selected. The selection of the step h is directly related to the order of the EPF $F(x)$ which will be used to mimic the original function $f(x)$. Let the order of $F(x)$ is m (we wish to mimic $f(x)$ using an EPF with the order equal to m). Then, the step h reads:

$$h = \frac{a-b}{2m-1}.$$

Now, the function $f(x)$ can be sampled at nodes of the grid:

$$p_0 = f(a);\, p_1 = f(a+h);\, p_2 = f(a+2h);\, \ldots;\, p_{2m-1} = f(b).$$

We have assumed that the minimal order of the EPF $F(x)$ is m. Therefore, according to (1) the following equality holds true:

$$d_{m+1} = \det \begin{bmatrix} p_0 & p_1 & \cdots & p_{m-1} & p_m \\ p_1 & p_2 & \cdots & p_m & p_{m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_m & p_{m+1} & \cdots & p_{2m-1} & F(b+h) \end{bmatrix} = 0. \tag{18}$$

Note that it is easy to determine $F(b+h)$ from (18) [8]. But we will not use $F(b+h)$ (nor $f(b+h)$) in further computations.

Now, the characteristic polynomial (2) takes the form:

$$\det \begin{bmatrix} f(a) & f(a+h) & \cdots & f(a+mh) \\ f(a+h) & f(a+2h) & \cdots & f(a+(m+1)h) \\ \cdots & \cdots & \cdots & \cdots \\ f(a+(m-1)h) & f(a+mh) & \cdots & f(b) \\ 1 & \rho & \cdots & \rho^m \end{bmatrix} = 0. \tag{19}$$

Let us assume that all roots $\rho_1, \rho_2, \ldots, \rho_m$ are different. Then, (14) can be used to compute indexes λ_r; $r = 1, 2, \ldots, m$. Now, a linear system of equations is constructed using (9); its solution produces coefficients μ_r; $r = 1, 2, \ldots, m$. Finally, the mimicking EPF in the interval $a \le x \le b$ reads:

$$F(x) = \sum_{r=1}^{m} \mu_r \exp(\lambda_r x).$$

If some roots of (19) are multiple, the algorithm of computations is similar, though the expression of the mimicking algebraic interpolant becomes more complex (4).

Example 5.1. *Algebraic interpolation of an EPF.*
Let us consider the following EPF:

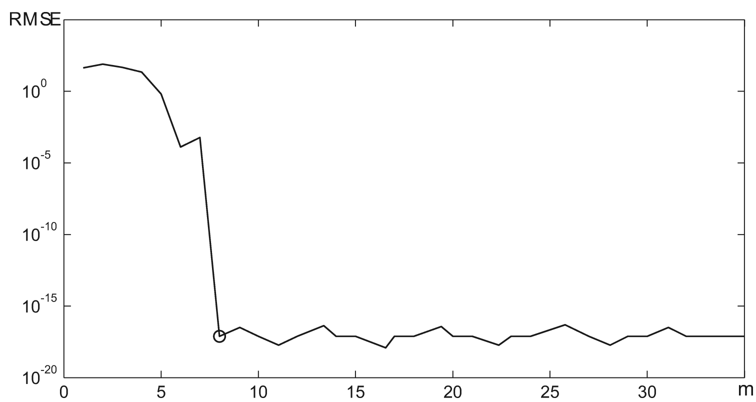
$$f_a(x) = 0.3x^2 \sin(2.14x)e^{-0.13x} + \cos(0.18x)e^{-0.31x}.$$

We will construct the algebraic interpolation of this function in the interval $0 \le x \le 10$. *Let us assume that the order of LRS of values of* $f_a(x)$ *is* m $(rank(p_j; j \in Z_0) = m)$; *the linear recurrent function is denoted as* $F_m(x)$. *Then,* $h = \frac{10}{2m-1}$ *and:*

$$p_j = f_a(jh); j = 0, 1, \ldots, (2m-1).$$

We perform a number of computational experiments for different values of m; $m = 1, 2, \ldots, 35$. *The algorithm of algebraic interpolation produces 35 different linear recurrent functions* $F_m(x)$ *and 35 values of RMSE (root mean square errors) of the interpolation defined as* $\sqrt{\frac{1}{10}\int_0^{10}(f_a(x) - F_m(x))^2\,dx}$. *The variation of RMSE from m is illustrated in Figure 3.*

Fig. 3. RMSE errors of the algebraic interpolation at different *m*; the circle denotes the best *m* (RMSE = 0 at m = 8).



It is clear that RMSE = 0 when $m = 8$ *because the order of* $f_a(x)$ *is equal to 8. We will illustrate the algorithm of algebraic interpolation for* $m = 8$ *in details. The step h is equal to* $\frac{2}{3}$, *then the characteristic polynomial has eight roots:* $\rho_{1,2,3} = 0.1317 + 0.9075i$; $\rho_{4,5,6} = 0.1317 - 0.9075i$; $\rho_{7,8} = 0.8074 \mp 0.0974i$. *Values of* λ_r; $r = \overline{1,8}$ *(computed using* (14) *at* $k_r = 0$; $r = \overline{1,8}$*) read:* $\lambda_1^* = \lambda_{1,2,3} = -0.0867 + 1.4267i$; $\lambda_2^* = \lambda_{4,5,6} = -0.0867 - 1.4267i$; $\lambda_{3,4}^* = \lambda_{7,8} = -0.2067 \mp 0.1200i$.

Now, (4) *yields the equality:*

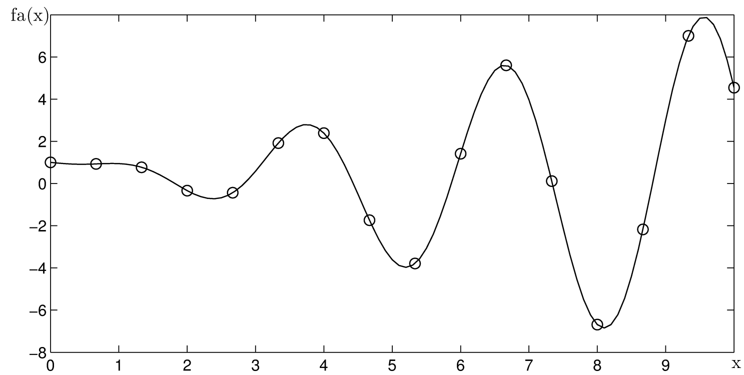
$$(\mu_1 + \mu_2 j + \mu_3 j^2)e^{j\lambda_1^*} + (\mu_4 + \mu_5 j + \mu_6 j^2)e^{j\lambda_2^*} + \mu_7 e^{j\lambda_3^*} + \mu_8 e^{j\lambda_4^*} = f(jh), j = \overline{0,7}.$$

Solutions of the linear algebraic system of equation now reads: $\mu_{1,2,4,5} = 0$; $\mu_{3,6} = \mp 0.0667i$; $\mu_{7,8} = \frac{1}{2}$.
Finally, the expression of $F_8(x)$ *reads:*

$$\begin{aligned}
F_8(x) &= \left(\frac{3}{2}\right)^2 (-0.0667i)x^2 e^{\frac{3}{2}(-0.0867+1.4267i)x} + \left(\frac{3}{2}\right)^2 (0.0667i)x^2 e^{\frac{3}{2}(-0.0867-1.4267i)x} \\
&+ \frac{1}{2}e^{\frac{3}{2}(-0.2067+0.12i)x} + \frac{1}{2}e^{\frac{3}{2}(-0.2067-0.12i)x} \\
&= -0.15ix^2 e^{-0.13x}(e^{2.14ix} - e^{-2.14ix}) + \frac{1}{2}e^{-0.31x}(e^{0.18ix} + e^{-0.18ix}) \\
&= -0.15ix^2 e^{-0.13x}(\cos(2.14x) + i\sin(2.14x) - \cos(-2.14x) - i\sin(-2.14x)) \\
&+ \frac{1}{2}e^{-0.31x}(\cos(0.18x) + i\sin(0.18x) + \cos(-0.18x) + i\sin(-0.18x)) \\
&= 0.3x^2\sin(2.14x)e^{-0.13x} + \cos(0.18x)e^{-0.31x}.
\end{aligned}$$

The algebraic interpolant is illustrated in Figure 4.

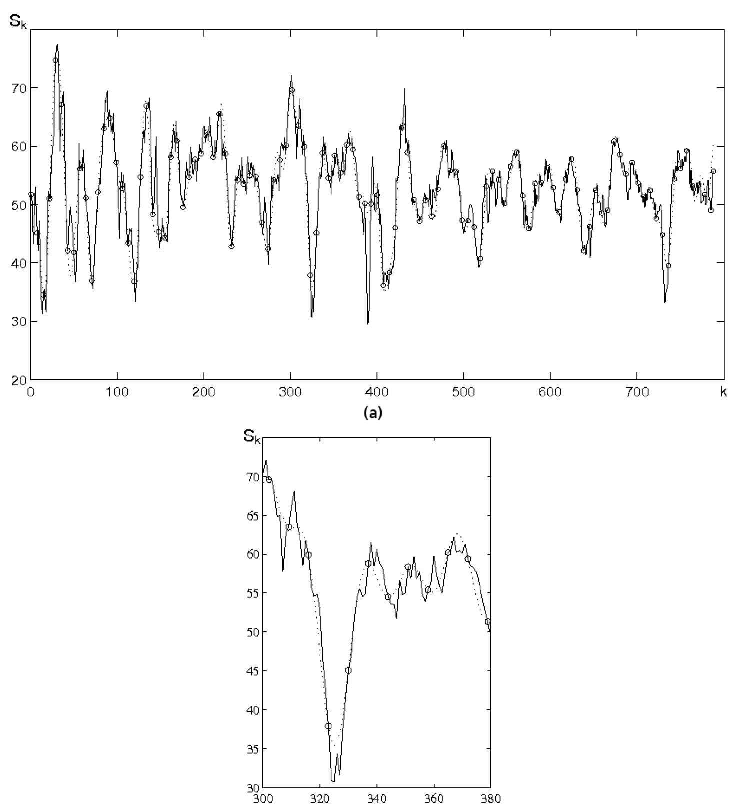
Fig. 4. Algebraic interpolant of $f_a(x)$ on interval $[0, 10]$ (the interpolant coincides with the function; circles denote nodes of the equispaced grid).



6 Algebraic interpolation of the real time series

The ability of algebraic interpolation on a regular grid (for the nearest algebraic interpolant) suggests interesting possibilities for application of this approximation scheme for the real time series. We use a time series of monthly PMI Composite Index. A PMI reading above 50 percent indicates that the manufacturing economy is generally expanding and below 50 percent that it is generally declining. The unfiltered PMI data S_k in interval $k = 1, 2, \ldots, 788$ is shown in Figure 5.

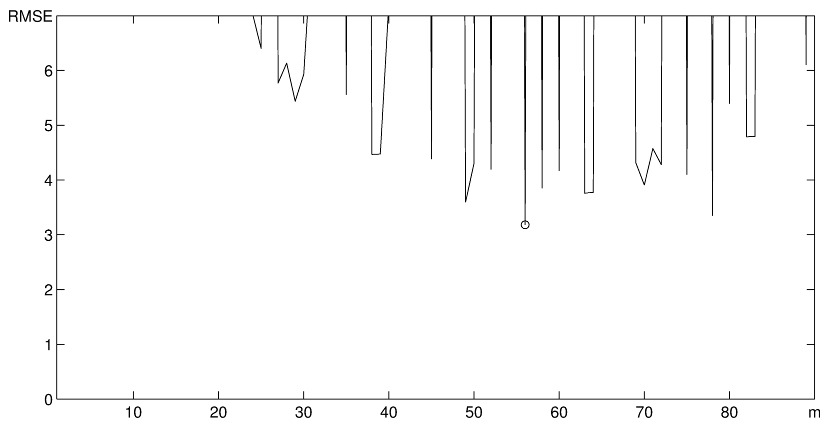
Fig. 5. PMI Composite Index data (the solid line) and an algebraic interpolant (the dotted line) of the experimental data at m = 56. Circles denote nodes of the equispaced grid; the zoomed part of the algebraic interpolant is illustrated in part (b).



We will use the algebraic interpolation scheme in equally spaced grid on interval [1; 788]. First, we preselect the order of the EPF which will be used to mimic the experimental data. For the order of the EPF equal to m, the step $h = \frac{787}{2m-1}$ and $p_n = S_j$; $j = 1 + nh$; $n = 0, 1, \dots, 2m-1$.

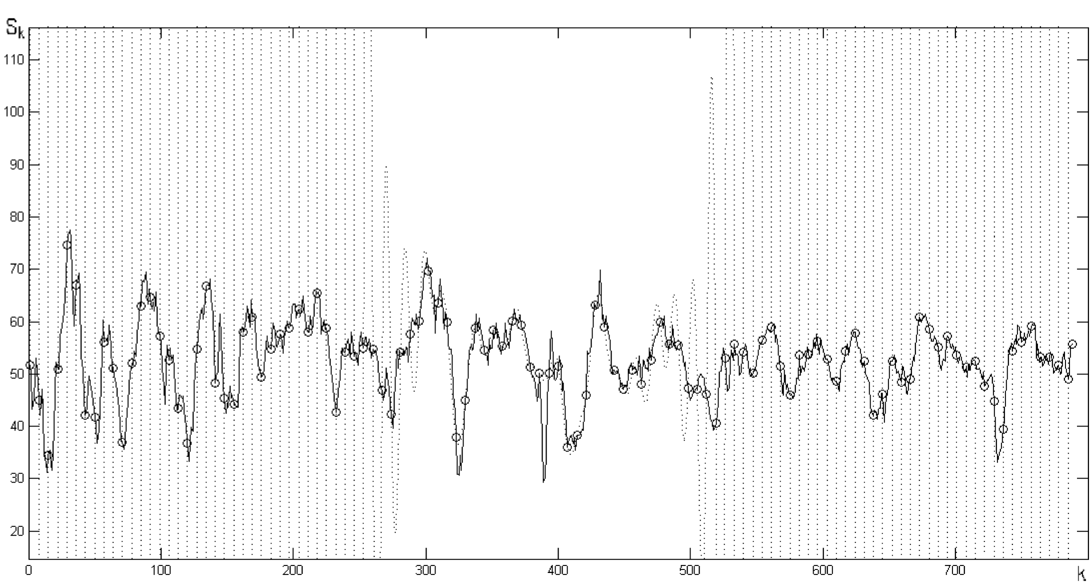
We again perform a number of computational experiments for different values of m; $m = 1, 2, \dots, 90$. The algorithm of algebraic interpolation produces 90 different EPF $F_m(x)$. RMSE of algebraic interpolation is now computed as the square root of the sum of squared differences between values of the given data and the values of the EPF $F_m(x)$ at all sampling points in the interval [1; 788]. Computational experiments show that the best result (the minimal RMSE = 3.17) is achieved at $m = 56$ (Figure 6). The graph of $F_{56}(x)$ is shown in Figure 5.

Fig. 6. RMSE errors of the algebraic interpolation at different m; the circle denotes the best m (RMSE = 3.17 at m = 56)



As noted previously, the extended Prony-type interpolation scheme outperforms the Lagrange polynomial on equispaced grinds. The Lagrangian interpolant is illustrated in Figure 7 (the nodes are the same as in Figure 5). Runge's effect prevents Lagrange interpolation to be a reasonable approximation (the maximum absolute value of the Lagrange interpolant in Figure 7 is equal to $3.1634 \cdot 10^{32}$), yet the proposed extended Prony-type interpolation does not have that problem.

Fig. 7. PMI Composite Index data (the solid line) and the Lagrangian interpolant (the dotted line). Circles denote nodes of the equispaced grid.



As mentioned in the Introduction, the polynomial interpolation in Chebyshev points is numerically stable even for high polynomial degrees. However, real-world time series are usually recorded using a constant sampling rate. Thus, it would be complicated to find values of this time series at Chebyshev points without transforming the scale (Chebyshev interpolation is straightforward for continuous functions of course). $F_{56}(x)$ in Figure 5 is a good example of such alternative interpolation.

7 Concluding remarks

An algebraic interpolation scheme on an equispaced grid is presented in this paper. It is demonstrated that the proposed scheme can be used for the identification of the nearest algebraic interpolant for a given function. This computational effect can be explained by the fact that though all nodes are located inside the bounded interval, the interpolant is reconstructed in the global domain. Such interpolation scheme can be extended to the extrapolation scheme what can be successfully exploited in time series prediction applications [8]. On the other hand, this advantageous feature can be successfully exploited for the analytic approximation of noisy and/or defected real-world signals.

Numerical experiments have shown that optimal interpolants produced by the proposed algebraic technique based on the order of LRS outperform classical Lagrange polynomial interpolants on equispaced grids. This effect can be explained by the fact that the functional base used to construct algebraic interpolants is wider compared to polynomial interpolants. Really, (8) would represent a polynomial interpolant if all indexes $\lambda_r; r = 0, 1, \ldots, n$ would be equal to zero.

The main drawback of the proposed interpolation scheme is that rather complex computations are required as the number of nodes becomes large. In the first place this is associated with the necessity to find all roots of the characteristic Hankel equation. Thus, the proposed scheme of interpolation looses its aptitude when the number of nodes becomes higher than one hundred. Nevertheless, the scheme preserves interesting potential of practical applications at lower number of nodes. The explicit error bound of the interpolation and the possibility of using adaptive grids remains a definite objective of future research.

Finally, it can be noted that the proposed scheme can be considered as an effective numerical tool for the identification of nearest algebraic "skeleton" functions and extends the applicability of classical interpolation schemes to real-world data contaminated with the inevitable noise.

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